Representing Periodic Functions in a Fourier Basis

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1 1D Sine and Cosine Transforms

Suppose we have a function $x(\theta)$ which is periodic in 2π. Suppose we want to find a new representation for $x(\theta)$, namely a Fourier series mapping new values of θ to a new value of x. The form of this Fourier series is

$$x(\theta) = \sum_{m=0}^{N_F} X_m^c \cos(m\theta) + X_m^s \sin(m\theta)$$

where the number of Fourier components tells us about the resolution of the function $x(\theta)$. To write $x(\theta)$ in this Fourier basis, we need to compute the coefficients X_m^c and X_m^s. We can do this using the inverse transforms

$$X_0^c = \frac{1}{N_\theta} \sum_{i=1}^{N_\theta} x(\theta_i)$$

$$X_0^s = 0$$

$$X_m^c = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} x(\theta_i) \cos(m\theta_i)$$

$$X_m^s = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} x(\theta_i) \sin(m\theta_i)$$

where $\theta_i = \frac{i}{2\pi N_\theta}$.

Note that the zeroth component of the Cosine component is just the average of $x(\theta)$ over the dataset.

1.1 Proof of inverse transform

Of course, we need to convince ourselves that the inverse transform I've written down is consistent with our representation of $x(\theta)$. To do this I'll plug in $x(\theta_i)$ and show that all the other terms vanish besides the coefficient. We start with X_0^c.

$$X_0^c = \frac{1}{N_\theta} \sum_{i=1}^{N_\theta} \sum_{m=0}^{N_F} X_m^c \cos\big(\frac{mi}{2\pi N_\theta}\big) + X_m^s \sin\big(\frac{mi}{2\pi N_\theta}\big)$$

The sum of Cosine and Sin over equally spaced intervals gives zero for all m except $m = 0$, where $\cos(0) = 1$ and $\sin(0) = 0$. Taking only the $m = 0$ term, we have

$$X_0^c = \frac{1}{N_\theta} \sum_{i=1}^{N_\theta} X_0^c = X_0^c$$

This proves that our inverse transform for X_0^c gives us the correct transform. Although I have chosen $X_0^s = 0$, since $\sin(0) = 0$, in reality any value of X_0^s could have been chosen since it plays no role in the transform.

Let's now show that our choices of X_m^c and X_m^s give us the correct inverse transforms.

$$X_m^c = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} \sum_{m'=0}^{N_F} X_{m'}^c \cos(m'\theta_i)\cos(m\theta_i) + \cos(m'\theta_i)\sin(m\theta_i)$$

$$X_m^s = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} \sum_{m'=0}^{N_F} X_{m'}^c \sin(m'\theta_i)\cos(m\theta_i) + \sin(m'\theta_i)\sin(m\theta_i)$$

Each of the terms where Sine multiplies Cosine vanishes in the sum over i. Each of the terms where Cosine multiplies Cosine and Sin multiplies Sine vanishes, unless $m = m'$, in which case the function averages to $1/2$.

$$X_m^c = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} \sum_{m'=0}^{N_F} X_{m'}^c \cos(m'\theta_i)\cos(m\theta_i)\delta_{m,m'} = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} \sum_{m'=0}^{N_F} X_m^c \frac{1}{2} = X_m^c$$

$$X_m^s = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} X_{m'}^c \sin(m'\theta_i)\sin(m\theta_i)\delta_{m,m'} = \frac{2}{N_\theta} \sum_{i=1}^{N_\theta} X_m^s \frac{1}{2} = X_m^s$$

This proves that we have chosen the correct inverse transforms.

2 2D Sine and Cosine Transforms

Now suppose we have a 2D function $x(\zeta, \theta)$ which is periodic in 2π in both variables. Suppose we want to find a new representation for $x(\zeta, \theta)$, namely a Fourier series mapping new values of (ζ, θ) to a new value of x. The form on this Fourier series is

$$x(\zeta, \theta) = \sum_{m=0}^{N_{F\zeta}} \sum_{n=0}^{N_{F\theta}} X^{cc}_{m,n} \cos(m\zeta) \cos(n\theta) + X^{cs}_{m,n} \cos(m\zeta) \sin(n\theta) + \\ X^{sc}_{m,n} \sin(m\zeta) \cos(n\theta) + X^{ss}_{m,n} \sin(m\zeta) \sin(n\theta)$$

To write $x(\zeta, \theta)$ in this Fourier basis, we need to compute the coefficients $X^{cc}_{m,n}$, $X^{cs}_{m,n}$, $X^{sc}_{m,n}$, and $X^{ss}_{m,n}$. We can do this using the inverse transforms

$$X^{cc}_{0,0} = \frac{1}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j)$$

$$X^{cs}_{0,0} = X^{sc}_{0,0} = X^{ss}_{0,0} = 0$$

$$X^{cc}_{0,n} = \frac{2}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \cos(n\theta_j)$$

$$X^{cs}_{0,n} = \frac{2}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \sin(n\theta_j)$$

$$X^{sc}_{0,n} = X^{ss}_{0,n} = 0$$

$$X^{cc}_{m,0} = \frac{2}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \cos(m\zeta_i)$$

$$X^{sc}_{m,0} = \frac{2}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \sin(m\zeta_i)$$

$$X^{cs}_{m,0} = X^{ss}_{m,0} = 0$$

$$X_{m,n}^{cc} = \frac{4}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \cos(m\zeta_i) \cos(n\theta_j)$$

$$X_{m,n}^{cs} = \frac{4}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \cos(m\zeta_i) \sin(n\theta_j)$$

$$X_{m,n}^{sc} = \frac{4}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \sin(m\zeta_i) \cos(n\theta_j)$$

$$X_{m,n}^{ss} = \frac{4}{N_\zeta N_\theta} \sum_{i=1}^{N_\zeta} \sum_{j=1}^{N_\theta} x(\zeta_i, \theta_j) \sin(m\zeta_i) \sin(n\theta_j)$$

where $\zeta_i = \frac{i}{2\pi N_\zeta}$ and $\theta_j = \frac{j}{2\pi N_\theta}$.

We can prove that these are the correct inverse transforms by plugging in our Fourier series for $x(\zeta, \theta)$ and using the orthogonality of Sines and Cosines like we did in the 1D case.