GENERAL PLASMA PHYSICS LECTURE NOTES

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ABSTRACT

These notes summarize and explain the topics covered in a graduate-level introductory plasma physics course at Princeton University (General Plasma Physics I, Fall 2017). I first define what a plasma is and introduce some of its basic properties. I then discuss single particle motion, kinetic and fluid models of plasmas, and plasma waves.

These notes were written during 2017-18. In early 2024, they were edited to improve clarity. If you are reading these notes and find a typo or error, please let me know via email at [first name][last name] AT gmail DOT com.

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1 Basics

It's unbelievable how much you don't know about the game you've been playing all your life.

MICKEY MANTLE

Plasma physics is a rich, varied subject. This richness comes mathematically, experimentally, as well as through the numerous applications of plasma physics research. Research in plasma physics draws knowledge from many areas of physics: electromagnetism, thermodynamics, statistical mechanics, nuclear physics, and atomic physics. Experiments in plasma physics often involve vacuum systems, superconducting coils, cryogenic systems, complex optical instruments, advanced materials for plasma-facing components, wave-guides, and much more. Computational plasma physics involves developing and implementing numerical algorithms, as well as linking computational work to physical models, theory, and experiment. It often uses some of the most powerful supercomputers in existence. Students of plasma physics are often interested in one or more applications. These include astrophysics, plasma thrusters, semiconductor manufacturing, and fusion energy.

These lecture notes focus on the theoretical foundations of the subject rather than concentrate on any particular application. They are based on an introductory plasma physics course at Princeton University, General Plasma Physics I, taught by Professors Nathaniel J. Fisch and Hong Qin in fall 2017.

1.1 Definition of a plasma

The simplest definition of a plasma is an ionized gas. But this simple definition leaves much to be desired. How ionized does it need to be to be a plasma? A gas of what?

Professor Fisch begins by pointing out that states of matter are really approximations of reality. Take, for example, a closed box stuffed with gravel. Each individual rock in that gravel behaves like a solid when we observe it. If we were to take that box and throw it in the air, it would rotate approximately like a solid body. But when we open the box and pour the gravel into a funnel, the behavior of the gravel is probably better described with a fluid approximation. Similarly, the tectonic plates which make up the earth's continents are solid when we look at them over the course of a day or a month or a year. But when we look at them over a timescale of millions of years, the plates travel, flow, and merge, unlike a solid.

His point is that whether or not some real physical system can be treated as one of the idealized states of matter depends on how we are observing that system. In the language of plasma physics, we would say that the state of matter some system is in depends on the the timescales and length scales which we are observing the system over. For example, in gas clouds in the interstellar medium the degree of ionization is very low and the magnetic fields are very small, but over large enough scales and over long enough times their evolution is well-described by the equations of plasma physics.

In some sense, plasmas fit somewhere along an energy spectrum, where the spectrum ranges over the energy per particle (i.e. temperature). At one end of the spectrum is condensed matter physics, i.e. solids. These are at the lowest temperature. As we increase the temperature, eventually the solids become fluids, fluids become gases, and at some point they become plasma-like. In the temperature range where gas becomes fully ionized, we have an ideal classical plasma ($\sim$10 eV to $\sim$100 KeV). If we were to turn up the temperature even further, then in the MeV range positrons start to become produced and we have a relativistic QED plasma. In this energy range, our equations of classical plasma physics are no longer valid and we have to develop other equations to understand this system. If we really crank up the energy dial, up to $\sim$ 100MeV, then we'll have a quark-gluon plasma, which is confined by the strong force rather than the Electromagnetic force. What we see from this discussion is that plasma physics is the physics of matter within a certain temperature range.

This still doesn't answer our question of "what is a plasma"! It turns out that this definition is a bit technical, but I'll state it here. Some system is a plasma if the number of plasma particles in a Debye sphere is much greater than 1, or $n_0 \frac{4}{3}\pi\lambda_D^3 \gg 1$. Often, this is just written as $n\lambda_D^3 \gg 1$. We define what the Debye length λ_D is in section 1.4. We will discuss this definition of a plasma in more depth in sections 1.6 and 3.3.2.

1.2 Logical framework of plasma physics

In some sense, plasma physics has been fully solved. A plasma consists of a large number of charged particles, typically ions and electrons. Suppose each of these particles has charge q and mass m. They interact via the Lorentz force,

$$m\frac{d^2\boldsymbol{r}}{dt^2} = q\left(\boldsymbol{E} + \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{B}\right) \tag{1.1}$$

The initial conditions for the electric and magnetic fields are given by two of Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{1.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{1.3}$$

while the time-evolution of the electric and magnetic fields are determined by the other two Maxwell equations,

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{1.5}$$

Given a set of initial conditions for the particles, we can solve for the time evolution of these particles. If we really wanted to be precise, we could even use the Lorentz-invariant force laws, and calculate the time-evolution of the plasma particles to arbitrary precision. Unfortunately, this simplistic approach doesn't work for a myriad of reasons, both *practical* and *physical*.

Practically, such an approach is not solvable analytically except with a few particles, and even using a computer to solve a macroscopic system with $O(10^{23})$ particles would be intractable. To make any progress in plasma physics, we obviously need a description of a plasma which can be practically solved. Thus, we will need to *approximate* somehow in order to get a tractable solution.

Physically, this simple model described in eqs. (1.1) and (1.5) is wrong, for at least two reasons. First, there is no consideration of boundary conditions. In any terrestrial plasma, the plasma will be confined to some region by a solid (or fluid) boundary, and the plasma particles will interact with the boundary in some complex way. Much of plasma physics research involves understanding the effects of plasmas as they interact with solids at the plasma boundary. In astrophysical plasmas, the boundaries are either ignored, not well-defined, or do not exist. In practice, periodic or open boundary conditions are often used to understand astrophysical plasmas. Second, particles do much more than just move under the action of the Lorentz force. They also combine, react, ionize, recombine, radiate, fuse, and otherwise act in ways not described by classical mechanics.

As you can imagine, a rigorous, complete description of a plasma would get extremely complicated pretty quickly. For this reason, approximation will be our friend as we study plasma physics. When applying approximation schemes (i.e., a model) to study plasmas, it is important to keep track of when the approximations we make are valid so as not to apply some equation to a physical situation where it is not applicable.

The main statistical (or kinetic) model in plasma physics involves using a statistically averaged distribution of particles to get a 6-dimensional (3 spatial dimensions, 3 velocity dimensions) time-evolving distribution function f. This distribution function f, called the Vlasov distribution, tells us the number of particles at a given position with a given velocity. We'll discuss kinetic theory in section 3.

Another model involves treating the plasma as a fluid. The multi-fluid model treats each plasma species[1] as a fluid with a well-defined density, velocity, and pressure at each point in space. Another model, called magnetohydrodynamics or MHD, treats the entire plasma as a single fluid. These fluid models of a plasma will be derived as an approximation of the kinetic model and discussed in section 4.

1.3 Plasma oscillations

The first phenomenon we'll study are plasma oscillations. Plasma oscillations illustrate many of the equations and techniques used in plasma physics and are the most simple example of what is called *collective dynamics*. Dynamics is the study of how a system evolves over time. Collective dynamics means that when interacting, plasma particles can conspire to create macroscopic dynamics which are different than what would be observed if the particles were not interacting. Plasma waves (section 5) and Landau damping are other examples of collective behavior. For now, let's look at a simplified model of plasma oscillations, where the ions are assumed to be motionless and only the electrons are allowed to move.

[1] Species means a category of particles - so ions, electrons, and neutral atoms are species. I use the subscript σ to refer to a particle of a particular species.

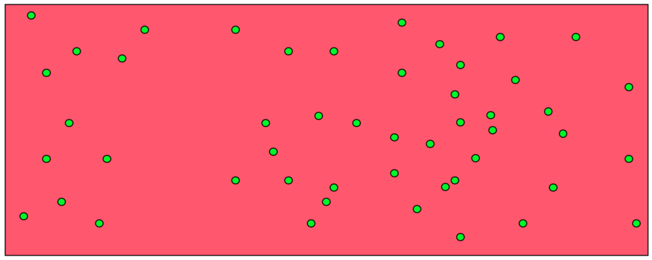


Figure 1: An initial electron density configuration. Perturbation is exaggerated for illustration.

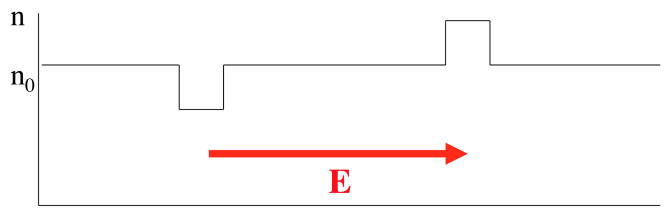


Figure 2: Electric field due to density perturbation. Over time, these bumps in density will rise and fall but stay at constant position in space in a cold plasma.

Suppose that, in an otherwise uniform plasma, some number of electrons are displaced to the right as in fig. 1. Since there is now a positive charge density to the left and a negative charge density to the right, an electric field is setup which points towards the right, as in fig. 2. The electrons between the two bumps will feel a force to the left, and will be accelerated leftwards. As they are accelerated leftwards, eventually $\boldsymbol{E} = 0$, but the electrons have developed a velocity leftwards. Their momentum carries them leftwards, which eventually creates a higher electron density on the left, and a lower electron density on the right. This process will repeat itself, and the net effect is that the density perturbations will oscillate in time but not in space. These oscillations are called plasma oscillations. Intuitively, plasma oscillations arise due to the electrostatic force which arises when electrons are displaced from an equilibrium. They are the result of the electrostatic restoring force in combination with electron inertia.

To derive these plasma oscillations, we start with the multi-fluid equations (derived in section 4). Instead of looking at individual particles, the fluid equations treat the density, velocity, etc, of the plasma as scalar and vector fields. For each species we have a continuity equation and a momentum equation. We'll also use Poisson's equation and assume that any electric fields are electrostatic ($\boldsymbol{E} = -\boldsymbol{\nabla}\phi$), and that the magnetic field is zero. We're looking for oscillations of the electrons, which we expect to be much faster than any oscillations of the ions because the electrons are much lighter.[2] We'll therefore assume that the ions are stationary ($\boldsymbol{u}_i = 0$) and have a constant and static density n_0. Our equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} e(n_0 - n_e) \tag{1.6}$$

$$\frac{\partial n_e}{\partial t} + \boldsymbol{\nabla} \cdot (n_e \boldsymbol{u}_e) = 0 \tag{1.7}$$

$$m_e n_e \frac{\partial \boldsymbol{u}_e}{\partial t} + m_e n_e (\boldsymbol{u}_e \cdot \boldsymbol{\nabla})(\boldsymbol{u}_e) = -e n_e \boldsymbol{E}. \tag{1.8}$$

We have obviously not derived these equations, but they are simple to understand. Equation (1.6) is the familiar Poisson's equation from electrostatics. Equation (1.7) is a continuity equation for electron density, similar to the charge conservation equation in electrodynamics. It just means that if the electron density inside a fixed infinitesimal volume changes in time, it is because there are electrons flowing across the boundary of that infinitesimal volume. Equation (1.8) is a momentum equation for the electrons. Essentially, it means that the mass times acceleration of electrons is equal to the force they feel due to the electric field.

For those who have seen fluid equations before, note that there is no pressure term in eq. (1.8). Pressure, as you will remember from elementary kinetic theory of gases, is an effect which comes about due to the microscopic motion

[2] In other words, the timescale at which the electrons oscillate will be much shorter than the timescale at which the ions oscillate. This differentiation of timescales between electrons and ions due to their different masses is a recurring theme throughout the study of plasma physics.

of molecules. Thus, whatever results we will derive are technically applicable only in the approximation of a zero-temperature (cold) plasma, where the molecules do not have thermal velocities. We'll see what happens in section 4 for plasmas with non-zero temperature.

We now introduce a method called linearization. Linearization is an important technique in plasma physics. For our purposes, linearization means the following: we take some equilibrium ($\frac{\partial}{\partial t} \to 0$) solution to the equations of interest, and call the equilibrium values the 0th order solution. We'll then assume that there is some (small) perturbation to the equilibrium solution. We call these perturbations the first-order quantities. We plug the first-order quantities into the equations and ignore any terms which are second-order or higher. This gives us linearized equations. We then look for solutions of the linearized equations.

Let's see linearization in action. For plasma oscillations, we start with the most basic equilibrium possible: a zero-velocity plasma ($\boldsymbol{u}_{e0} = \boldsymbol{u}_{i0} = 0$), with a uniform density of electrons and ions ($n_{e0}(\boldsymbol{r}) = n_{i0}(\boldsymbol{r}) = n_0$) and zero electric field ($\phi_0 = \text{constant}$). Then, we apply a small perturbation to all relevant quantities, except ion density and ion velocity which we've assumed to be constant over the timescales we're interested in. This gives $\boldsymbol{u}_e = \boldsymbol{u}_1$, $n_e = n_0 + n_{e1}$, and $\phi = \phi_1$. By ignoring all terms second-order or higher, we have

$$\boldsymbol{\nabla}^2 \phi_1 = -\frac{e}{\epsilon_0}\Big(n_0 - n_0 - n_{e1}\Big) = e n_{e1}/\epsilon_0 \tag{1.9}$$

$$\frac{\partial n_{e1}}{\partial t} = -\boldsymbol{\nabla} \cdot (n_0 \boldsymbol{u}_1) = -n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u_1} \tag{1.10}$$

$$m_e n_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = e n_0 \boldsymbol{\nabla} \phi_1 \tag{1.11}$$

Now taking the divergence of the linearized momentum equation (eq. (1.11)), we have

$$m_e n_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_1}{\partial t} = -m_e \frac{\partial^2 n_{e1}}{\partial t^2} = e n_0 \boldsymbol{\nabla}^2 \phi_1 = \frac{e^2 n_0}{\epsilon_0} n_{e1}$$

$$\frac{\partial^2 n_{e1}}{\partial t^2} = -\omega_p^2 n_{e1}(\boldsymbol{r}, t) \tag{1.12}$$

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e}. \tag{1.13}$$

Equation (1.12) gives an oscillatory solution for the electron density perturbation n_{e1}. Because the derivative in eq. (1.12) is a partial derivative with respect to time, the density perturbation oscillates in time but not in space. This means that the density perturbations in fig. 2 oscillate up and down in time with period $T = 2\pi/\omega_p$, but do not change their shape. This is the same physical situation we described earlier. Note that the electron fluid velocity and ϕ oscillate in time as well.

1.4 Debye shielding

As we remember from electromagnetism, the electric field inside a conductor is zero. Otherwise charges would move around, modifying the electric field until it becomes zero.

Plasmas, in general, are highly conducting. Thus, we should expect that the electric field inside a plasma is 0, right? Well, not exactly. Indeed plasmas, like conductors, screen external electric fields quite well. However, due to the thermal velocities of charged particles, the electric field inside a plasma is not necessarily zero. It turns out that if we place a charge Ze in a plasma and make it stay there, then in equilibrium the electric potential a distance r away from the charge is

$$\phi = \frac{Ze}{4\pi\epsilon_0 r} e^{-r/\lambda_D} \tag{1.14}$$

where λ_D is a constant called the Debye length which depends on, among other things, temperature. This faster-than-exponential falloff of the electric potential is what is called Debye shielding or Debye screening. Over distances longer than a few Debye lengths λ_D, the electric potential due to a charge in the plasma becomes very small. Loosely speaking, plasmas are net neutral over distances longer than a Debye length.

Physically, Debye shielding is an effect which arises due to the balance between electric forces and random thermal velocities. If we put a test charge in a plasma at a fixed location, the other particles will feel a force due to that charge. If the plasma particles had zero thermal velocity, they would move towards or away from the charge until they had no electric force on them, and eventually settle into an equilibrium where the forces on all particles are zero. However, because particles have thermal velocity, they don't stay at rest but instead fly about randomly. As a result, the electric potential around a test charge in a plasma isn't completely shielded as it would be in an ideal conductor. The Debye length is the length scale over which a plasma shields electric fields. Based on this picture, we expect a higher-temperature plasma to have a larger Debye length because the random motion will be faster. We also expect a plasma made up of particles with larger charge q_σ to have a smaller Debye length, because the particles will feel a stronger electrostatic force from any test charge.

Let's derive eq. (1.14), the electric potential around a test charge $+Ze$ in a plasma. Imagine inserting a test particle of infinitesimal charge Q into a plasma. Assume that each species (represented by the subscript σ) in the plasma is in thermal equilibrium with temperature T_σ, and that each species can be treated as a fluid with density n_σ. Now, it turns out that in thermal equilibrium

$$n_\sigma = n_0 e^{\frac{-q_\sigma \phi}{k_B T_\sigma}}. \tag{1.15}$$

This is simply the Boltzmann distribution of statistical mechanics, where the energy level $E = q_\sigma \phi$. We can derive this Boltzmann relation from the fluid equations, assuming thermal equilibrium. We use the equation of motion

$$m_\sigma \frac{d\boldsymbol{u}_\sigma}{dt} = q_\sigma \boldsymbol{E} - \frac{1}{n_\sigma}\boldsymbol{\nabla} P_\sigma. \tag{1.16}$$

Assuming that the inertial term on the left-hand side (LHS) is negligible (meaning the changes in the plasma are slow), the electric field is electrostatic ($\boldsymbol{E} = -\boldsymbol{\nabla}\phi$), the temperature is spatially uniform, and the ideal gas law $P_\sigma = n_\sigma k_B T_\sigma$ holds, then this reduces to

$$0 = -n_\sigma q_\sigma \boldsymbol{\nabla}\phi - k_B T_\sigma \boldsymbol{\nabla} n_\sigma \tag{1.17}$$

which has the solution

$$n_\sigma = n_0 e^{-q_\sigma \phi / k_B T_\sigma}. \tag{1.18}$$

The assumptions we just made are all consistent with the plasma being in thermal equilibrium, which is the assumption used to derive the Boltzmann distribution from statistical mechanics. So our work checks out. Now, we assume that $k_B T_\sigma \gg q_\sigma \phi$, which is true when the test particle's charge is small so that the ϕ created by this charge is small as well.[3] Taylor expanding eq. (1.18) in this limit, we get

$$n_\sigma \approx n_0 \Big(1 - \frac{q_\sigma \phi}{k_B T_\sigma}\Big) \tag{1.19}$$

Now that we have the density of species σ as a function of ϕ, we can use Poisson's equation to solve for the electric potential from a test charge. Assuming the test charge is at the origin, Poisson's equation gives us

$$-\boldsymbol{\nabla}^2 \phi = \frac{1}{\epsilon_0}\Big(Q\delta^{(3)}(\boldsymbol{r}) + \sum_\sigma n_\sigma(\boldsymbol{r}) q_\sigma\Big). \tag{1.20}$$

Using eq. (1.19) and the fact that the plasma is net neutral to zeroth order, this simplifies to

$$-\boldsymbol{\nabla}^2 \phi + \frac{\phi}{\lambda_D^2} = \frac{1}{\epsilon_0} Q\delta^{(3)}(\boldsymbol{r}) \tag{1.21}$$

where

$$\frac{1}{\lambda_D^2} = \sum_\sigma \frac{1}{\lambda_{D\sigma}^2}, \tag{1.22}$$

$$\lambda_{D\sigma} = \sqrt{\frac{\epsilon_0 k_B T_\sigma}{q_\sigma^2 n_0}}. \tag{1.23}$$

Now, $\boldsymbol{\nabla}^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \phi}{\partial r})$. We look for a solution of the form

$$\phi = \frac{f(r)Q}{4\pi\epsilon_0 r} \tag{1.24}$$

where $f(0) = 1$. This is an inspired guess, which gives us the potential for a test charge in the $r \to 0$ limit but diverges from the typical $\frac{1}{r}$ dependence as $r > 0$. Plugging this in to eq. (1.21), we have

$$-\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 \frac{\partial}{\partial r}\Big(\frac{f(r)}{r}\Big)\Big) + \frac{f(r)Q}{4\pi\epsilon_0 r \lambda_D^2} = \frac{1}{\epsilon_0} Q\delta^{(3)}(\boldsymbol{r}). \tag{1.25}$$

[3] This doesn't works in the $r \to 0$ limit, because the field of the test charge ϕ goes as $\frac{1}{r}$. So technically eq. (1.14), which we are deriving now, breaks down very close to our charge, and becomes a better approximation as we get further away from the charge.

This first term is a bit tricky to simplify. From the chain rule, we can see that we'll have four terms when we expand. One of the terms will be $-\frac{Q}{4\pi\epsilon_0}f(r)\boldsymbol{\nabla}^2\frac{1}{r}$, a second will be $-\frac{Q}{4\pi\epsilon_0 r}f''(r)$, and the third and fourth will be cross-terms, proportional to $f'(r)$. It turns out that the cross-terms will cancel each other, as we see below.

$$\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big(\frac{f(r)}{r}\Big)\Big) = \frac{\partial}{\partial r}\Big(r^2\Big(\frac{f'(r)}{r} - \frac{f(r)}{r^2}\Big)\Big) = f'(r) - f'(r) + \text{other terms}$$

With the cancellation of the cross-terms, eq. (1.25) becomes

$$-\frac{Qf(r)}{4\pi\epsilon_0}\boldsymbol{\nabla}^2\frac{1}{r} - \frac{Q}{4\pi\epsilon_0 r}f''(r) + \frac{f(r)Q}{4\pi\epsilon_0 r\lambda_D^2} = \frac{1}{\epsilon_0}Q\delta^{(3)}(\boldsymbol{r}) \tag{1.26}$$

Remembering from electromagnetism that $\boldsymbol{\nabla}^2(\frac{1}{r}) = -4\pi\delta^{(3)}(\boldsymbol{r})$, the first term on the left hand size (LHS) and the right hand side (RHS) cancel. This leaves us with

$$f'' = \frac{f}{\lambda_D^2} \tag{1.27}$$

which has the solution

$$f(r) = e^{-r/\lambda_D} \tag{1.28}$$

The positive exponential solution is ruled out due to boundary conditions at $r = \infty$; the potential at infinity can't be infinity. Plugging $f(r)$ into our inspired guess from eq. (1.24), we have the Debye shielding equation for a test charge Q in a plasma,

$$\phi(r) = \frac{Q}{4\pi\epsilon_0 r}e^{-r/\lambda_D}. \tag{1.29}$$

Note that the Debye length is larger for larger values of T_σ, and smaller for larger values of q_σ. This makes sense, since Debye screening is an effect we see due to the thermal motion of charged particles in tandem with the electrostatic forces they feel. Loosely speaking, a species' charge causes it to want to stay close to any test charge Q in the plasma, thus large q_σ should decrease the Debye length, by increasing the electrostatic force on these particles. On the other hand, a species' thermal motion causes it to zoom around randomly, so large T_σ should increase the Debye length by increasing these random speeds. A zero temperature plasma has zero Debye length, because (in equilibrium) the particles will have no thermal velocity and thus exactly cancel the potential due to any test charges. Note also that the Debye length *doesn't* depend on the particle's mass. While low-mass particles (i.e. electrons) will have larger thermal energy, they will also be accelerated more easily the electric forces. It turns out that the factors of m_σ cancel in giving us our Debye length. Note also the following nifty little relation, neglecting a numerical factor of $\sqrt{2}$ or $\sqrt{3}$ in the thermal velocity $v_{\text{th}\sigma}$.

$$\lambda_{D\sigma} = \frac{v_{\text{th}\sigma}}{\omega_{p\sigma}} = \sqrt{\frac{k_B T_\sigma}{m_\sigma}}\sqrt{\frac{m_\sigma\epsilon_0}{q_\sigma^2 n_0}} \tag{1.30}$$

We can actually use the physical intuition we've developed to remember this in a slightly different way. The plasma frequency $\omega_{P\sigma}$ comes about as the interaction between electric forces and particle inertia. The thermal velocity $v_{\text{th}\sigma}$ is, roughly, the balance between particle thermal energy and particle inertia. The Debye length $\lambda_{D\sigma}$ is the balance between particle thermal energy and electric forces. We can write this, roughly, as

$$\lambda_{D\sigma} = \frac{\text{Thermal energy}}{\text{Electric forces}} = \frac{\frac{\text{Thermal energy}}{\text{Inertia}}}{\frac{\text{Electric forces}}{\text{Inertia}}} = \frac{v_{\text{th}\sigma}}{\omega_{P\sigma}}$$

I find remembering which factors go where in the Debye length and the plasma frequency can be tricky, unless I'm using some sort of physical intuition like this to help me figure it out.

Here's something seemingly contradictory that confused me: the electric field inside a plasma is not always 0! But we learned in freshman physics that the electric field inside a conductor is 0. We also know that plasmas are highly conducting. So what is it about a plasma which is different than a typical conductor, such as a metal? Actually, in terms of shielding of electric fields, nothing. In an idealized metal, the electrons are at a temperature T and are free to move around as they please. Their behavior obeys Poisson's equation and the Boltzmann relation. Therefore, we must see Debye shielding in a metal! In fact, if we were to put a test charge in a metal and hold it there, we would see a potential quite like the Debye potential. At the edge of a charged conductor, where there is a surface charge, the electric field will not go to 0 immediately inside the conductor! Instead, it will fall to from σ/ϵ_0 to 0 over a couple Debye lengths.

1.5 Collisions in plasmas

Collisions can be a difficult, subtle, yet important topic in plasma physics. The Princeton course Irreversible Processes in Plasmas (AST554) discusses collisions in much more depth [6]. For now, we'll just consider a few basic ideas.

1.5.1 Brief tutorial

The distance of closest approach between two particles in a collision is approximately the distance at which the average kinetic energy equals the electrostatic potential energy $\frac{1}{4\pi\epsilon_0}\frac{q^2}{b}$, where b is the distance of closest approach. This would occur if a particle with kinetic energy $\frac{3}{2}k_B T$ were moving directly towards a stationary particle, converting all the kinetic energy into potential energy. Solving for b, we get

$$b = \frac{q^2}{6\pi\epsilon_0 k_B T} \tag{1.31}$$

The collision cross-section is roughly $\sigma = \pi b^2$, so

$$\sigma = \frac{q^4}{36\pi\epsilon_0^2 (k_B T)^2} \tag{1.32}$$

The mean free path is defined as

$$l = \frac{1}{\sigma n}. \tag{1.33}$$

This brief discussion captures the general scaling of b, σ, and l.

1.5.2 Small- versus large-angle collisions

In a plasma where the number of particles in a Debye sphere is much greater than 1, small-angle (grazing) collisions dominate large-angle collisions. Why is this? Essentially, it comes down to the fact that the coulomb force is a long-range force. In neutral gases, where we don't have the Coulomb interaction between particles, short-range large-angle collisions are most important. In charged gases, the long-range interactions end up being most important. In a plasma, the electric force between two charged particles goes like $\sim -\partial/\partial r\left(e^{-r/\lambda_D}/r\right)$. This faster-than exponential falloff of the electric field means that in a plasma, the force between two charged particles is large enough to cause a small-angle deflection until about $r \sim \lambda_D$. If the number of charged particles in a Debye sphere is much greater than 1, then there will be *many* grazing (small-angle) collisions between particles. The cumulative effects of these many grazing collisions at large distances, as calculated in both [1] and [6], is greater than the effects of the few large-angle collisions at small distances.

1.5.3 Collision timescales

The ideas in this subsection are discussed in more detail in chapter 1 of [1].

Plasmas are made up of (at least) two species of charged particles, electrons and (at least one species of) ions. Each particle carries momentum and energy. If every species is in thermodynamic equilibrium and has the same net momentum and the same average energy (i.e., the same temperature), then the system is in a maximum-entropy state and collisions will not transfer net momentum or energy between species. However, if one or more species is out of thermodynamic equilibrium or has net momentum or different average energy relative to the rest of the plasma, then collisions will transfer net momentum and/or energy between species in a way that increases entropy.

We define two frequencies at which particles transfer momentum and energy: the momentum collision frequency ν, and the energy collision frequency ν_E. Because frequency has units of s^{-1}, the timescale of each process is given by $1/\nu$ or $1/\nu_E$. $\nu_{\sigma\alpha}$ is defined as the frequency for which species σ transfers all of its momentum to species α.[4] Similarly, $\nu_{E\sigma\alpha}$ is defined as the frequency for which species σ transfers all of its energy to species α.

The enormous difference in masses between electrons and ions ($m_p/m_e \approx 1840$, $\sqrt{m_p/m_e} \approx 40$) means that electron-electron collisions are very different from ion-electron collisions, both of which are very different from ion-ion collisions. Because electrons are so much less massive, they move more quickly and their momentum and energy can be exchanged more easily. We'd therefore expect that the electron-electron collision frequencies would be much larger than ion-ion collision frequencies. The relative size of the collision frequencies between ions and electrons is summarized in table 1.

[4] In practice, this means the inverse time to scatter a particle by 90 degrees due to small-angle deflections.

~ 1	$\sim (m_e/m_i)^{1/2}$	$\sim m_e/m_i$
ν_{ee}	ν_{ii}	ν_{ie}
ν_{ei}	ν_{Eii}	ν_{Eei}
ν_{Eee}		ν_{Eie}

Table 1: Momentum and energy collision frequencies for like-particle and unlike-particle collisions between ions and electrons.

Physical intuition

We now attempt to develop physical intuition to understand each of the entries in table 1.

In table 1, we see that $\nu_{ee}/\nu_{Eee} \sim 1$ and $\nu_{ii}/\nu_{Eii} \sim 1$. In other words, for like-particle collisions the momentum scattering frequency is the same as the energy transfer frequency. Intuitively, this is because when a particle collides elastically with a particle of the same species, in the frame where one of the particles is at rest, we have (in 1-D, using conservation of momentum and energy from freshman physics) that $\Delta v_1 = v_1 = \Delta v_2$. In that frame, all of the momentum and energy of the first particle is transferred to the second particle. This suggests that for each species, there is a single frequency at which that species transfers all of its momentum and energy to itself. As we see in table 1, the electron-electron collision frequencies are a factor $(m_e/m_i)^{1/2} \sim 40$ higher than the ion-ion collision frequencies. This is because the thermal velocity of electrons is higher by a factor of $(m_e/m_i)^{1/2}$, causing electrons to collide with one another more frequently than do ions.

For energy transfer between ions and electrons, we see that ions transfer their energy to electrons as fast as electrons transfer their energy to ions. The energy transfer rate, however, is extremely slow: a factor of $m_p/m_e \approx 1840$ lower than ν_{ee}. The energy transfer rate is so low because when electrons and ions collide, the velocities of each species don't change much. To see this, consider a 1D collision between an ion and electron in the frame where the ion starts at rest. Using conservation of momentum and energy from freshman physics, the ion velocity after the collision is about $2(m_e/m_i)v_e$ and the ion energy (in this frame) is $m_i v_i^2/2 = 2m_e^2 v_e^2/m_i$. It will take on order of m_i/m_e collisions for the original electron kinetic energy ${}^{m_e v_e^2}\!/_2$ to be imparted to the ion, and vice versa.

For momentum transfer between ions and electrons, we see that ν_{ei} is a factor m_i/m_e larger than ν_{ie}. In other words, electrons transfer all of their momentum to ions much faster than ions transfer all of their momentum to electrons. To understand why, imagine that in 1D an electron is moving with some velocity v_e relative to a stationary ion. After the collision, once again using conservation of momentum and energy, the electron has a velocity slightly larger than $-v_e$. In other words, it only takes one collision for all of the electron momentum to be imparted to ions, implying that the momentum transfer frequency from electrons to ions ν_{ei} is the same as ν_{ee}. Now imagine that in 1D an ion is moving with some velocity v_i relative to a stationary electron. After the collision, the ion velocity is approximately $(1 - 2m_e/m_i)v_i$, only slightly lower than the original velocity. It will take on the order of m_i/m_e collisions with an electron before an ion has imparted all of its momentum to electrons, implying that $\nu_{ie}/\nu_{ei} \sim m_e/m_i$.

Examples

Let's consider a few examples to illustrate the physical meaning of these collision frequencies.

Suppose that at $t = 0$ there is a plasma in which the ions are initially cold and at rest, while the electrons are initially hot and have some net velocity $\boldsymbol{u}_0$. How does the plasma evolve? The fastest timescale is given by the first column of table 1. On this timescale, the electrons collide with the stationary ions, transferring their momentum. At a time on the order of $\sim \nu_{ei}^{-1}$, the ions and electrons both have a net velocity that scales like $\frac{m_e}{m_i}\boldsymbol{u}_0$. The next changes occur over the longest timescale, given by the third column of table 1. At a time on the order of ν_{Eei}^{-1}, the thermal energy of the electrons will be transferred to the ions, until eventually the temperature reaches an equilibrium of $(T_{e0} + T_{i0})/2$.

Suppose instead that we view the same situation from a different frame of reference. In this frame, the electrons are initially hot and at rest, while the ions are cold and have a net velocity $-\boldsymbol{u}_0$. Since this is just the same situation but viewed in a different frame of reference, then of course the same physics must take place. The energy equilibrium again takes place over the long timescale of ν_{Eei}^{-1}. This time, however, we have ions transferring momentum to electrons. It seems like that should happen on the ν_{ie}^{-1} timescale, instead of the ν_{ei}^{-1} timescale, right? Not exactly. ν_{ie} is the frequency at which ions transfer *all* of their momentum to electrons. Here, ions only need to transfer a tiny fraction $\frac{m_e}{m_i}$ of their momentum to the electrons for the electrons to have the same velocity as the ions. After a time $\frac{m_e}{m_i}\nu_{ie}^{-1} \sim \nu_{ei}^{-1}$, the ions have given the electrons a net velocity $-\boldsymbol{u}_0(1 - \frac{m_e}{m_i})$, while their velocity has decreased slightly to this value as well. If we transfer back to the frame of reference of the previous example, this gives us the same results, as it must.

Now suppose that at $t = 0$ there is a plasma in which the ions begin completely stationary, half of the electrons begin at rest, and the other half begin with velocity $\boldsymbol{u}_0$. On the ν_{ee}^{-1} timescale, the electrons with initial velocity $\boldsymbol{u}_0$ will scatter off the other electrons, and be deflected in velocity space. On the ν_{ei}^{-1} timescale, which is of the same order as ν_{ee}^{-1}, the electrons will transfer all of their momentum to the ions. It only takes a single collision for electrons to transfer their momentum to ions. If the electrons *weren't* colliding with ions, then they would have velocity $\boldsymbol{u}_0/2$ at this point. However, they are giving up all of their momentum to ions on this timescale, so the total momentum of the plasma needs to be conserved. This means that after a time of order ν_{ei}^{-1}, the electrons and ions will have the *same* net velocity, which from conservation of momentum has to be $\boldsymbol{u}_0(\frac{m_e}{m_e+m_i})$.[5] The ions are already in thermal equilibrium and momentum equilibrium with each other, so nothing changes on the ν_{ii}^{-1} timescale. On the very long ν_{Eei}^{-1} timescale, the electrons and ions thermalize, and approach the same temperature as usual.

Now suppose that at $t = 0$ there is a plasma in which the electrons are at rest, half of the ions begin at rest, and the other half begin with velocity $\boldsymbol{u}_0$. This example is a bit trickier. The ions transfer all of their momentum to the electrons on the very long ν_{ie}^{-1} timescale. However, the ions only need to transfer a small fraction of their momentum to the electrons in order for the two species to have the same net velocity. This means that the electrons will have the same net velocity as the ions on the ν_{ei}^{-1} timescale. Because there are two populations of ions, one at rest and one with velocity $\boldsymbol{u}_0$, the electrons will reach a velocity between the two on this fast timescale, or about $\boldsymbol{u}_0/2$. On the slower ν_{ii}^{-1} timescale, the ions will transfer momentum and energy between the two populations and reach a velocity set by conservation of momentum, a bit less than $\boldsymbol{u}_0/2$. Once again, on the very long ν_{Eei}^{-1} timescale the electrons and ions will reach thermal equilibrium with each other.

1.6 Plasma length and time scales

Length scales

There are numerous length scales in plasmas. Some important length scales include the distance of closest approach

$$b = \frac{e^2}{6\pi\epsilon_0 k_B T}, \tag{1.34}$$

the interparticle spacing

$$n^{-1/3}, \tag{1.35}$$

the mean free path

$$\lambda_{mfp} = \frac{1}{n\pi b^2}, \tag{1.36}$$

the electron gyro-radius

$$\rho_e = \frac{m_e v_{\text{the}}}{eB} = \frac{\sqrt{k_B T_e m_e}}{eB}, \tag{1.37}$$

the ion gyro-radius

$$\rho_i = \frac{\sqrt{k_B T_i m_i}}{ZeB}, \tag{1.38}$$

and the Debye length

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n_0}}. \tag{1.39}$$

The electron and ion gyro-radius size depends on the local magnetic field, which can vary dramatically between different plasmas. The ion gyro-radius is nearly always significantly larger than the electron gyro-radius, as long as the electron temperature is not dramatically larger than the ion temperature.

In a plasma where the number of particles in a Debye sphere is much greater than 1, we have the following ordering of scale lengths:

- $b \ll n^{-1/3} \ll \lambda_D \ll \lambda_{mfp}$

This is true so long as the plasma has a large number of particles in a Debye sphere. We now show that this is true. To start, define Λ as the number of particles in a debye sphere, $\Lambda = {}^{4\pi}/_{3} n\lambda_D^3$ and assume that $\Lambda \gg 1$. We have that

$$\lambda_D^2 = \frac{\epsilon_0 k_B T}{e^2 n_0} = \frac{1}{6\pi n_0 b} \tag{1.40}$$

[5]Section 1.9 of [1] considers the same example, but reaches a slightly different conclusion. One of us is incorrect.

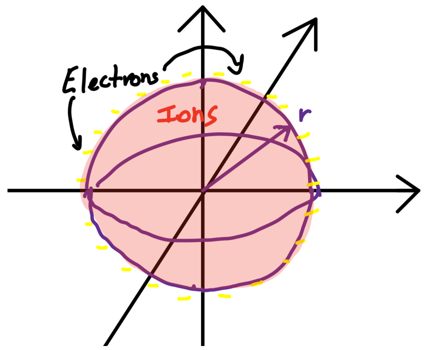


Figure 3: Illustration of the electrons (in yellow, on the surface of the sphere) evacuating a region of space with radius r, leaving a density of ions (in red) behind. This calculation is used to show why the number of particles in a Debye sphere being much greater than 1 implies overall quasineutrality over distances larger than a Debye length.

$$\frac{b}{\lambda_D} = \frac{1}{6\pi n_0 \lambda_D^3} = O(1/\Lambda). \tag{1.41}$$

λ_D is bigger than the distance of closest approach b by a factor of order Λ. We also have that

$$\frac{n^{-1/3}}{b} = \frac{n^{-1/3}}{\lambda_D}\frac{\lambda_D}{b} = O(\Lambda^{-1/3})O(\Lambda) = O(\Lambda^{2/3}) \tag{1.42}$$

The interparticle spacing $n^{-1/3}$ is bigger than the distance of closest approach by a factor of $\Lambda^{2/3}$. Finally, we have that

$$\frac{\lambda_{mfp}}{\lambda_D} = \frac{1}{n\pi b^2 \lambda_D} = \frac{\lambda_D^2}{\pi b^2}\frac{1}{n\lambda_D^3} = O(\Lambda^2)O(1/\Lambda) = O(\Lambda). \tag{1.43}$$

Putting it all together, we have (in units where $b = 1$), $b : 1$, $n^{-1/3} : \Lambda^{2/3}$, $\lambda_D : \Lambda$, $\lambda_{mfp} : \Lambda^2$, which gives our ordering of scale lengths in a plasma. We know the mean free path in a plasma is much longer than the Debye length, which is much larger than the interparticle spacing, which is much larger than the distance of closest approach. This ordering of scale lengths suggests that a plasma is a sparse, low-collisionality gas of interacting particles. I should emphasize again that all of this relies on the fact that the number of particles in a Debye sphere, Λ, is much greater than 1.

We can see now why we chose our earlier definition of a plasma to be where the number of particles in a Debye sphere is much greater than 1. If we choose this definition, then we have a definite ordering of scale lengths which gives qualitatively similar behavior for a variety of different plasma environments.

If $\Lambda \gg 1$, then the plasma is quasineutral. Quasineutral means almost neutral, i.e., for a two-species plasma $(q_e n_e - q_i n_i)/n_e \ll 1$. We can prove quasineutrality using a clever calculation. Imagine that all of the electrons in some region of space were to move radially outwards from a point until their velocity becomes zero, as in fig. 3. How large of a spherical region could the electrons evacuate, leaving only positively charged ions behind? This can be determined by setting the thermal energy of the electrons inside the volume (which all end up at the surface of the sphere) equal to the energy stored in the electromagnetic field created by ions left behind in the absence of the electrons. From Gauss's law

$$E_r 4\pi r^2 = \frac{ne4\pi r^3}{3\epsilon_0}$$
$$E_r = \frac{ner}{3\epsilon_0}.$$

The electromagnetic field energy is

$$\int \frac{\epsilon_0}{2} E^2 dV = \frac{2\pi n^2 e^2}{9\epsilon_0}\int r'^4 dr' = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0}.$$

The thermal energy per particle is $\frac{3}{2}k_B T$, so the total thermal energy of the electrons in that volume is $2\pi n k_B T r_{max}^3$. Setting the two energies equal, we have

$$2\pi n k_B T r_{max}^3 = \frac{2\pi n^2 e^2 r_{max}^5}{45\epsilon_0}$$

so the maximum radius r_{max} that the thermal energy of the electrons could evacuate is

$$r_{max} = \sqrt{\frac{45\epsilon_0 k_B T}{ne^2}} \approx 7\lambda_D \tag{1.44}$$

If the number of particles in a Debye sphere of volume ${}^{4\pi}/_{3}\lambda_D^3$ were less than 1, then the probability that the velocities of all particles inside a sphere of radius ${}^{4\pi}/_{3}(7\lambda_D)^3$ were moving approximately radially might be non-negligible. In that case, the probability that a plasma would be non-quasineutral over distances larger than a Debye length would be non-negligible. However, if the number of particles in a Debye sphere is much greater than 1, then the probability that due to random thermal motion all of the particles inside that sphere would be moving outwards would be extremely small. In that case, the plasma would be extremely likely to stay quasineutral when measured over distances larger than a Debye length.

Time scales

Some of the most important frequencies in plasma physics include the electron gyro-frequency

$$\Omega_e = \frac{eB}{m_e}, \tag{1.45}$$

the ion gyro-frequency

$$\Omega_i = \frac{q_i B}{m_i}, \tag{1.46}$$

the electron plasma frequency

$$\omega_{pe} = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_e}}, \tag{1.47}$$

and the electron-ion momentum collision frequency ν_{ei}. The associated timescales are the inverse of the frequencies.

2 Single particle motion

I therefore don't have much to say. But I will talk a long time anyway.

RICHARD FEYNMAN

As we've seen, the mean free path of particles in a plasma is significantly longer than any of the other scale lengths, so long as the number of particles in a Debye sphere is much greater than 1. Particles often travel a long distance before colliding with other particles. For many plasmas, the collision timescale ν^{-1} is much longer than other relevant timescales. In magnetized plasmas, an ion or electron might $\boldsymbol{v} \times \boldsymbol{B}$ rotate (sometimes called gyro-motion or Larmor motion) many times before it collides with another plasma particle, changing its trajectory. Thus, analyzing the motion of individual charged particles gives valuable insight into the behavior of the plasma as a whole.

We will first investigate the motion of particles in prescribed electric and magnetic fields. We'll see that four main particle drifts show up: the $\boldsymbol{E} \times \boldsymbol{B}$ drift, the $\boldsymbol{\nabla} \boldsymbol{B}$ drift, the curvature drift, and the polarization drift. We will also see that periodic motion in slowly changing fields leads to the existence of conserved quantities for individual particles, which can be helpful for analyzing the motion of particles in complicated electromagnetic fields. These conserved quantities are called adiabatic invariants. We'll look at two adiabatic invariants, μ and $\mathcal{J}$. We will then analyze the magnetic mirror machine, the classic example of single-particle motion. Lastly, we'll discuss the isorotation theorem, an example of single-particle motion not typically found in textbooks but which Nat covered in class.

2.1 Guiding center drifts

Imagine we have a constant, static magnetic field in the z-direction, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$. If we put a charged particle of charge q and mass m in that magnetic field, then the particle will move in a spiral in the plane perpendicular to the magnetic field while its velocity in the z-direction will remain constant. Let's show this. The force on the particle is $q\boldsymbol{v} \times \boldsymbol{B}$, which always points perpendicular to the motion. As we know from freshman physics, if the force is always perpendicular to the velocity then we have uniform circular motion. Thus, we have a centripetal acceleration $v_\perp^2/R = qv_\perp B_0/m$. This is easily solved, as in freshman physics, to give a frequency $\Omega = qB_0/m$ and a gyro-radius $\rho = \frac{mv_\perp}{qB_0}$.

In many plasmas, there exists some sort of uniform background magnetic field. Usually this background magnetic field is created by external magnetic coils, internal current, or a background field in astrophysics. The most basic, ubiquitous behavior of single particles in a plasma is gyro-motion around this background magnetic field. However, the behavior of particles in spatially-varying and time-varying electromagnetic fields is much more complicated. We will see that the guiding center motion (i.e., the center of the orbit) involves slow drifts across the magnetic field in addition to fast gyro-orbits around the magnetic field and streaming along the magnetic field.

Suppose there exists a charged particle of mass m and charge q in arbitrary electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. The equation of motion for the charged particle is

$$\ddot{\boldsymbol{r}} = \frac{q}{m}\Big(\boldsymbol{E}(\boldsymbol{r}, t) + \dot{\boldsymbol{r}} \times \boldsymbol{B}(\boldsymbol{r}, t)\Big) \tag{2.1}$$

Let us assume that $\boldsymbol{E}$ and $\boldsymbol{B}$ are known. In general, this expression cannot be integrated exactly to solve for the motion. However, we will make a few approximations for this problem to become solvable. Firstly, we will assume that the particle gyro-orbits around the magnetic field, and that the gyro-radius of the particle is small relative to the length scales ($\frac{B}{\boldsymbol{\nabla} B}$) over which the electric and magnetic fields change. Thus, $\boldsymbol{r}(t) = \boldsymbol{r}_{gc}(t) + \boldsymbol{\rho}(t)$ where $\boldsymbol{r}_{gc}$ is the position of the guiding center of the particle, and $\boldsymbol{\rho}$ is the vector from the guiding center to the particles position. This is illustrated in fig. 4. We will see that there are a number of drifts of $\boldsymbol{r}_{gc}$ which add to each other in the limit that the gyro-radius is much smaller than the relevant lengthscales of the magnetic and electric fields.

We define $\boldsymbol{\rho}$ as $\frac{m\hat{\boldsymbol{b}} \times \dot{\boldsymbol{r}}}{qB}$, where the magnetic field is evaluated at the position of the guiding center. Note that this definition makes sense intuitively. Remember when we had a constant magnetic field, such that the gyro-radius ρ was $\frac{mv_\perp}{qB}$? Notice that our definition here is essentially the same - $\boldsymbol{\rho}$ is perpendicular to both $\hat{\boldsymbol{b}}$ and $\boldsymbol{v}$, points in the right direction, and reduces to our previous expression in the limit that the magnetic field is constant in space.

2.1.1 E cross B drift

The first guiding center drift we will examine is called the E cross B drift. We know that if there is an electric field parallel to the local magnetic field, a particle will accelerate in that direction without feeling any magnetic force.

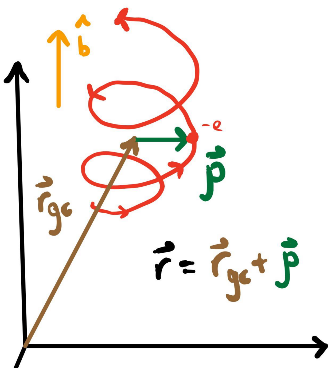


Figure 4: A negatively charged particle moving in a magnetic field. The motion of the particle is shown in red. The gyro-radius $\boldsymbol{\rho}$ is shown in green, and the guiding center position $\boldsymbol{r}_{gc}$ is shown in brown.

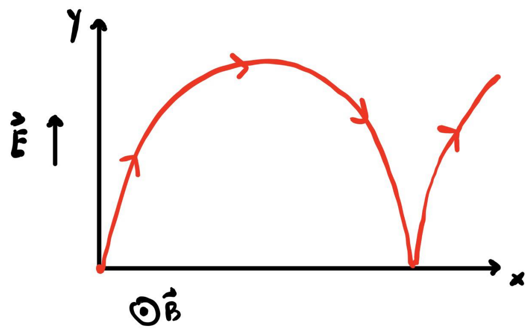


Figure 5: Illustration of the $\boldsymbol{E} \times \boldsymbol{B}$ drift for a particle which starts at rest at the origin. For particles with other initial velocities, the motion will have a different trajectory but the same drift velocity of the guiding center.

However, electric fields perpendicular to the local magnetic field direction give rise to the $\boldsymbol{E} \times \boldsymbol{B}$ drift

$$\boldsymbol{v}_{E\times B} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}. \tag{2.2}$$

Physical intuition

Imagine that there is a static constant $\boldsymbol{B}$ field in the z-direction and a static constant $\boldsymbol{E}$ field in the y-direction. Now imagine at $t = 0$ putting a positive charge $+q$ at the origin with zero initial velocity. The electric field causes the charge to initially accelerate in the y-direction. As the charge picks up speed in the y-direction, the magnetic field puts a force in the x-direction on the charge, causing it to turn in the positive x-direction. As the particle turns, eventually its velocity is entirely in the x-direction. Now, the magnetic force will be in the negative y-direction, and it turns out that this force will be stronger than the electric force in the positive y-direction. Thus, the particle starts to curve downwards, in the negative y-direction. At some point, the particle will return to $y = 0$, and then the process will repeat itself. However, the particle will have been displaced in the x-direction, which is also the $\boldsymbol{E} \times \boldsymbol{B}$ direction. This process is illustrated in fig. 5. Negatively charged particles (i.e. electrons) will initially accelerate downwards in fig. 5, but the magnetic force will cause them to curve towards the right in figure 5, again creating a drift in the $\boldsymbol{E} \times \boldsymbol{B}$ direction.

Derivation

For simplicity, we will assume that the magnetic field is constant over the gyro-orbit of the particle. We have

$$\boldsymbol{r}_{gc}(t) = \boldsymbol{r} - \boldsymbol{\rho} = \boldsymbol{r} - \frac{m\hat{\boldsymbol{b}} \times \dot{\boldsymbol{r}}}{qB}.$$

Taking the time-derivative, we have

$$\dot{\boldsymbol{r}}_{gc} = \dot{\boldsymbol{r}} - \frac{m\hat{\boldsymbol{b}}}{qB} \times \ddot{\boldsymbol{r}}.$$

Inserting the equation of motion eq. (2.1) into $\ddot{\boldsymbol{r}}$ gives

$$\dot{\boldsymbol{r}}_{gc} = \dot{\boldsymbol{r}} - \frac{\hat{\boldsymbol{b}} \times \boldsymbol{E}}{B} - \hat{\boldsymbol{b}} \times (\dot{\boldsymbol{r}} \times \hat{\boldsymbol{b}}). \tag{2.3}$$

The rightmost term is $\dot{\boldsymbol{r}}_\perp$, the velocity perpendicular to the local magnetic field. We also know that $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_\parallel + \dot{\boldsymbol{r}}_\perp$. The two $\dot{\boldsymbol{r}}_\perp$s cancel and we are left with

$$\dot{\boldsymbol{r}}_{gc} = \dot{\boldsymbol{r}}_\parallel + \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}. \tag{2.4}$$

This perpendicular drift of the guiding center is the E cross B drift we described earlier. Notice that if we replaced $q\boldsymbol{E}$ with an arbitrary force $\boldsymbol{F}$, we would get a drift velocity

$$\boldsymbol{v}_{\boldsymbol{F}} = \frac{\boldsymbol{F} \times \boldsymbol{B}}{qB^2}. \tag{2.5}$$

For example, this force could be the force of gravity, $\boldsymbol{F} = m\boldsymbol{g}$. In laboratory plasmas, this equation tells us that gravity causes positive and negative particles to drift in opposite directions, creating a current.[6] This effect is very small, so in general we can neglect it.

2.1.2 Grad-B drift

The grad-B drift is an effect that arises due to gradients in the magnetic field strength in the direction perpendicular to the magnetic field. The grad-B drift is equal to

$$\boldsymbol{v}_{\boldsymbol{\nabla} B} = \frac{v_\perp}{2} \frac{\rho \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{B^2} = \frac{m v_\perp^2}{2} \frac{\boldsymbol{B} \times \boldsymbol{\nabla} B}{qB^3}. \tag{2.6}$$

Physical intuition

This drift arises because particles have smaller gyro-radii in regions of stronger magnetic field, and larger gyro-radii in regions of weaker magnetic field. This is illustrated in figs. 6a and 6b.

Derivation

Suppose that a straight magnetic field which points in the $\hat{\boldsymbol{b}}$ direction everywhere has a gradient, as illustrated in fig. 6. We'd like to derive the drift velocity of a charged particle $\boldsymbol{v}_{\boldsymbol{\nabla} B}$ to lowest order in $\epsilon = \rho/L$ where L is the scale length of the magnetic field gradient.

We'll start with the expression

$$\int_0^{2\pi} \boldsymbol{\rho} d\theta = \hat{\boldsymbol{b}} \times \Delta \boldsymbol{r} \tag{2.7}$$

where $\Delta \boldsymbol{r}$ is the distance the guiding center travels in one rotation along the magnetic field. To derive eq. (2.7), use the definition of the gyro-radius

$$\boldsymbol{\rho} = \frac{m\hat{\boldsymbol{b}} \times \dot{\boldsymbol{r}}}{qB} = \frac{m\hat{\boldsymbol{b}}}{qB} \times \frac{d\boldsymbol{r}}{dt}.$$

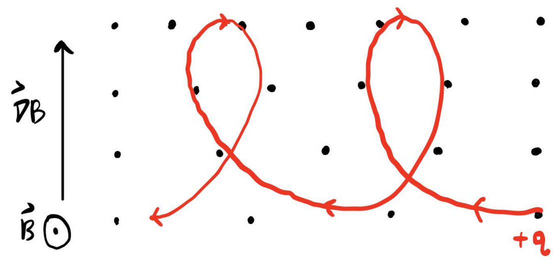
Performing the integral of $\boldsymbol{\rho}$ with respect to θ gives

$$\int_0^{2\pi} \boldsymbol{\rho} d\theta = \hat{\boldsymbol{b}} \times \int \frac{m}{qB(\boldsymbol{r})} \frac{d\boldsymbol{r}}{dt} d\theta = \hat{\boldsymbol{b}} \times \int \frac{m}{qB(\boldsymbol{r})} \frac{d\boldsymbol{r}}{d\theta} \Big(\frac{d\theta}{dt} \Big) d\theta. \tag{2.8}$$

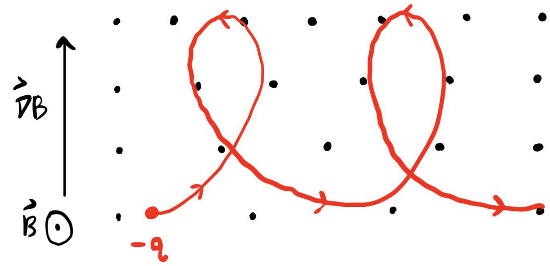
Using $\frac{d\theta}{dt} = \Omega = \frac{qB}{m}$, the factors of Ω cancel and we are left with

$$\int_0^{2\pi} \boldsymbol{\rho} d\theta = \hat{\boldsymbol{b}} \times \int \frac{d\boldsymbol{r}}{d\theta} d\theta = \hat{\boldsymbol{b}} \times \Delta \boldsymbol{r}. \tag{2.9}$$

[6] [5] has a nice discussion of the gravitational drift. Basically, in a finite plasma (let's say a tokamak), the gravitational current points in the horizontal direction and will lead to a buildup of charge on the side walls of a machine. This buildup of charge leads to an electric field in the horizontal direction, creating a $\boldsymbol{E} \times \boldsymbol{B}$ drift of both species downwards. This is the same thing that happens in a tokamak without a poloidal magnetic field - the grad-B and curvature drifts cause particles to drift in opposite directions vertically, creating a vertical electric field which causes particles to drift out of the tokamak.



(a) Positive charge



(b) Negative charge

Figure 6: Illustration of the grad-B drift of charged particles in a magnetic field gradient.

We'd like to calculate $\Delta \boldsymbol{r}$, because this allows us to calculate the drift velocity using

$$\boldsymbol{v}_{\boldsymbol{\nabla} B} = \Delta \boldsymbol{r}/T = \frac{\Delta \boldsymbol{r}\Omega}{2\pi} \tag{2.10}$$

where T is the period of a gyro-orbit $\frac{2\pi}{\Omega}$. Calculating $\boldsymbol{\Delta} r$ requires $\boldsymbol{\rho}$ to first-order in ϵ:

$$\boldsymbol{\rho}(\boldsymbol{r}) = \frac{m}{q}\frac{\hat{\boldsymbol{b}} \times \dot{\boldsymbol{r}}}{B(\boldsymbol{r}_{gc} + \boldsymbol{\rho})} \approx \frac{m}{qB(\boldsymbol{r}_{gc})}(\hat{\boldsymbol{b}} \times \dot{\boldsymbol{\rho}})\Big(1 - \frac{(\boldsymbol{\rho}\cdot\boldsymbol{\nabla})B}{B(\boldsymbol{r}_{gc})}\Big). \tag{2.11}$$

Plugging eq. (2.11) into eq. (2.7) gives

$$\hat{\boldsymbol{b}} \times \Delta \boldsymbol{r} = \int \frac{m}{qB(\boldsymbol{r}_{gc})}(\hat{\boldsymbol{b}} \times \dot{\boldsymbol{\rho}})\Big(1 - \frac{(\boldsymbol{\rho}\cdot\boldsymbol{\nabla})B}{B(\boldsymbol{r}_{gc})}\Big)d\theta. \tag{2.12}$$

Next we need $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ as a function of θ. Assume we have a positive particle and we set our coordinate system to point along the local magnetic field, such that $\boldsymbol{B} = B_z(\boldsymbol{r})\hat{\boldsymbol{z}}$. To 0th order, $\boldsymbol{\rho}(\theta) = \rho(\cos\theta\hat{\boldsymbol{x}} - \sin\theta\hat{\boldsymbol{y}})$. Similarly, $\dot{\boldsymbol{\rho}} = v_\perp(-\sin\theta\hat{\boldsymbol{x}} - \cos\theta\hat{\boldsymbol{y}})$. We plug these into eq. (2.12) and perform the integration over θ. The first term in eq. (2.12) integrates to 0 because it is linear in $\sin\theta$ and $\cos\theta$. The second term, which is non-linear in $\sin\theta$ and $\cos\theta$, is non-zero. Let's solve for it:

$$(\boldsymbol{\rho}\cdot\boldsymbol{\nabla})B = \rho\cos\theta\frac{\partial B_z}{\partial x} - \rho\sin\theta\frac{\partial B_z}{\partial y}$$

$$-(\hat{\boldsymbol{b}} \times \dot{\boldsymbol{\rho}})(\boldsymbol{\rho}\cdot\boldsymbol{\nabla})B =$$

$$v_\perp\rho\hat{\boldsymbol{b}} \times \Bigg[\Big(\sin\theta\cos\theta\frac{\partial B_z}{\partial x} - \sin^2\theta\frac{\partial B_z}{\partial y}\Big)\hat{\boldsymbol{x}} + \Big(\cos^2\theta\frac{\partial B_z}{\partial x} - \sin\theta\cos\theta\frac{\partial B_z}{\partial y}\Big)\hat{\boldsymbol{y}}\Bigg].$$

We're going to integrate this expression over 2π. The first and fourth terms of this expression integrate to 0, because they go like $\sin\theta\cos\theta$. $\sin^2\theta$ and $\cos^2\theta$ integrated over 2π give π. This leaves us with

$$\hat{\boldsymbol{b}} \times \Delta \boldsymbol{r} = \int \rho d\theta = \frac{\pi m\rho v_\perp}{qB^2}\hat{\boldsymbol{b}} \times (-\frac{\partial B_z}{\partial y}\hat{\boldsymbol{x}} + \frac{\partial B_z}{\partial x}\hat{\boldsymbol{y}}). \tag{2.13}$$

We're getting close. We can see pretty easily that

$$-\frac{\partial B_z}{\partial y}\hat{\boldsymbol{x}} + \frac{\partial B_z}{\partial x}\hat{\boldsymbol{y}} = \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B$$

so from eq. (2.13) this becomes

$$\Delta \boldsymbol{r} = \frac{\pi m \rho v_\perp}{qB^2} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B. \tag{2.14}$$

Plugging this into eq. (2.10) gives our desired result, eq. (2.6).

2.1.3 Curvature drift

As we know, in straight constant magnetic fields charged particles stream freely parallel to the magnetic field and gyrate perpendicular to the magnetic field, following field lines. Remarkably, in curved magnetic fields, charged particles *also* follow field lines, at least to lowest order in ϵ.

However, charged particles do not perfectly follow curved field lines; to first order in ϵ they drift across the magnetic field. The drift due to curvature of the magnetic field is called the curvature drift. It is given by

$$\boldsymbol{v}_c = \frac{m v_\parallel^2}{qB} \hat{\boldsymbol{b}} \times (\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{b}} \tag{2.15}$$

and it arises whenever $\hat{\boldsymbol{b}}$ changes.

Physical intuition

[7] has a very short derivation of the curvature drift that gives us some insight into the physical intuition for the curvature drift. Imagine that a particle is traveling along a field line with velocity $v_\parallel$ in a curved, constant-strength magnetic field. In the rotating frame of the particle, there is a centrifugal pseudo-force in the radial direction equal to $\boldsymbol{F} = m v_\parallel^2 / R \hat{\boldsymbol{r}}$. Plugging this force into eq. (2.5) for the $\boldsymbol{F} \times \boldsymbol{B}$ drift gives a drift velocity of

$$\boldsymbol{v}_D = \frac{m v_\parallel^2}{qB} \frac{\hat{\boldsymbol{r}}}{R} \times \hat{\boldsymbol{b}}. \tag{2.16}$$

Using $(\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{b}} = (\frac{1}{R} \frac{d}{d\phi}) \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{r}}/R$, we see that this is identical to eq. (2.15).

The physical intuition for the curvature drift isn't as clear as for the other drifts. However, this derivation suggests a simple physical picture. As a charged particle travels parallel to a curved magnetic field, it 'feels' (in its frame of reference) a centrifugal force in the radially outwards direction. Initially, this causes the particle to move radially outwards, in the same way that the electric field in the $\boldsymbol{E} \times \boldsymbol{B}$ drift initially causes particles to move in the direction of the electric field. As the particle accelerates radially outwards, the $q\boldsymbol{v}_r \times \boldsymbol{B}$ force causes the particle's velocity to change either upwards or downwards, depending on the sign. The dynamics of the motion are now nearly the same as in the $\boldsymbol{E} \times \boldsymbol{B}$ drift of fig. 5, except the displacement is now vertical. In summary, particles begin to accelerate radially due to the centrifugal force, and the magnetic field turns this radial velocity into vertical displacement.

Derivation

We now derive the curvature drift more formally, as was done in class. We assume that a charged particle travels through a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ that varies in space and that $\boldsymbol{E} = 0$. Our strategy is to expand the equation of motion eq. (2.1) to lowest order in $\epsilon = \rho/L$, average over the gyro-orbit, then solve for the time-averaged perpendicular velocity of the guiding center $\langle \dot{\boldsymbol{r}}_{gc,\perp} \rangle$.

We once again choose $\boldsymbol{r} = \boldsymbol{r}_{gc} + \boldsymbol{\rho}$. We then expand $\boldsymbol{B}$ around the gyro-center, such that $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r_{gc}}) + (\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{B}$. Thus eq. (2.1) becomes

$$\ddot{\boldsymbol{r}}_{gc} + \ddot{\boldsymbol{\rho}} = \frac{q}{m} \Big[\Big(\dot{\boldsymbol{r}}_{gc} + \dot{\boldsymbol{\rho}} \Big) \times \Big(\boldsymbol{B}(\boldsymbol{r}_{gc}) + (\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{B}(\boldsymbol{r}_{gc}) \Big) \Big]. \tag{2.17}$$

In the limit $\frac{\rho}{L} = \epsilon \ll 1$, we can define the gyro-motion to be the solution to the equation

$$\ddot{\boldsymbol{\rho}} = \frac{q}{m} \dot{\boldsymbol{\rho}} \times \boldsymbol{B}(\boldsymbol{r}_{gc}). \tag{2.18}$$

Subtracting eq. (2.18) from eq. (2.17) gives

$$\ddot{\boldsymbol{r}}_{gc} = \frac{q}{m} \Big[\dot{\boldsymbol{r}}_{gc} \times \boldsymbol{B}(\boldsymbol{r}_{gc}) + \dot{\boldsymbol{\rho}} \times (\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{B}(\boldsymbol{r}_{gc}) + \dot{\boldsymbol{r}}_{gc} \times (\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{B}(\boldsymbol{r}_{gc}) \Big]. \tag{2.19}$$

Now we average this equation over one gyro-period. The third term, because it is linear in $\boldsymbol{\rho}$, will integrate to 0 to first order in ϵ. The averaged first and second terms give

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle = \frac{q}{m} \Big[\langle \dot{\boldsymbol{r}}_{gc} \rangle \times \boldsymbol{B}(\boldsymbol{r}_{gc}) + \langle \dot{\boldsymbol{\rho}} \times (\boldsymbol{\rho} \cdot \boldsymbol{\nabla}) \boldsymbol{B}(\boldsymbol{r}_{gc}) \rangle \Big]. \tag{2.20}$$

The first term on the right hand side (RHS) ends up contributing to the curvature drift, while the second term on the RHS end ups contributing to the $\boldsymbol{\nabla} B$ drift. Since we've already calculated the grad-B drift, we won't calculate the second term, but it equals $\frac{-\mu\boldsymbol{\nabla}B}{m}$ where $\mu = \frac{mv_\perp^2}{2B}$.[7] This gives

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle = \frac{q}{m}(\langle \dot{\boldsymbol{r}}_{gc} \rangle \times \boldsymbol{B}(\boldsymbol{r}_{gc})) + \frac{-\mu \boldsymbol{\nabla} B}{m} \tag{2.21}$$

To calculate the first term, take the cross product of eq. (2.21) with $\hat{\boldsymbol{b}}$ which gives

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle \times \hat{\boldsymbol{b}} = \frac{qB}{m}(\langle \dot{\boldsymbol{r}}_{gc} \rangle \times \hat{\boldsymbol{b}}) \times \hat{\boldsymbol{b}} + \frac{\mu \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{m}. \tag{2.22}$$

The first term simplifies to $-\frac{qB}{m}\langle \dot{\boldsymbol{r}}_{gc,\perp} \rangle$. Solving for $\langle \dot{\boldsymbol{r}}_{gc,\perp} \rangle$ gives

$$\langle \dot{\boldsymbol{r}}_{gc,\perp} \rangle = \frac{\mu \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{qB} - \frac{m\langle \ddot{\boldsymbol{r}}_{gc} \rangle \times \hat{\boldsymbol{b}}}{qB}. \tag{2.23}$$

The first term is the $\boldsymbol{\nabla} B$ drift. Simplifying the second term requires solving for $\ddot{\boldsymbol{r}}_{gc}$, which to first-order in ϵ is

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle = \frac{d}{dt}\dot{\boldsymbol{r}}_{gc} = \frac{d}{dt}(v_\parallel \hat{\boldsymbol{b}} + \boldsymbol{v}_{drift}) + O(\epsilon^2) = \frac{dv_\parallel}{dt}\hat{\boldsymbol{b}} + v_\parallel \frac{d\hat{\boldsymbol{b}}}{dt} + O(\epsilon^2). \tag{2.24}$$

Because the magnetic field changes slowly in time and space, then $\frac{d\boldsymbol{v}_{drift}}{dt}$ is at least second order in ϵ. Because $\boldsymbol{E} = 0$ then $\frac{dv_\parallel}{dt} = 0$ is at least second-order in epsilon as well. So

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle = v_\parallel \frac{d\hat{\boldsymbol{b}}}{dt} + O(\epsilon^2). \tag{2.25}$$

If s is the distance along a field line, then

$$\frac{d\hat{\boldsymbol{b}}}{dt} = \frac{\partial \hat{\boldsymbol{b}}}{\partial s}\frac{\partial s}{\partial t} = v_\parallel \frac{\partial \hat{\boldsymbol{b}}}{\partial s}. \tag{2.26}$$

It also turns out that

$$\frac{\partial \hat{\boldsymbol{b}}}{\partial s} = (\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{b}} \tag{2.27}$$

which can be understood geometrically by looking at a point in space where the magnetic field is instantaneously in the z-direction. In that case $\frac{\partial \hat{\boldsymbol{b}}}{\partial s} = \frac{\partial \hat{\boldsymbol{b}}}{\partial z}$, and $(\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{b}} = (\frac{\partial}{\partial z})\hat{\boldsymbol{b}}$. Equation (2.27) implies that

$$\frac{d\hat{\boldsymbol{b}}}{dt} = v_\parallel (\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{b}}$$

so to first order in ϵ

$$\langle \ddot{\boldsymbol{r}}_{gc} \rangle = v_\parallel^2 (\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{b}}. \tag{2.28}$$

Plugging this into eq. (2.23) gives

$$\langle \dot{\boldsymbol{r}}_{gc,\perp} \rangle = \frac{\mu \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{qB} + \frac{mv_\parallel^2}{qB}\hat{\boldsymbol{b}} \times (\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{b}} \tag{2.29}$$

The second term is the curvature drift (eq. (2.15)) while the first term is the grad-B drift (eq. (2.11)).

2.1.4 Polarization drift

The polarization drift is a drift that arises due to a time-dependent drift velocity $\boldsymbol{v}_{drift}$. However, if the time-variation in $\boldsymbol{v}_{drift}$ is mostly due to a time-dependent $v_{\boldsymbol{E}\times\boldsymbol{B}}$ drift, then the polarization drift is given by

$$\boldsymbol{v}_p = \frac{d\boldsymbol{E}_\perp}{dt}\frac{m}{qB^2}. \tag{2.30}$$

The polarization drift gives rise to a polarization current, similar to the polarization current in a dielectric medium.

[7] To perform this calculation, use the zeroth order solution to the gyro-motion, $\boldsymbol{\rho} = \cos\theta\hat{\boldsymbol{x}} - \sin\theta\hat{\boldsymbol{y}}$, and plug it in to this term.

Physical intuition

[3] explains the intuition behind the polarization drift:

> The physical reason for the polarization current is simple. Consider an ion at rest in a magnetic field. If a field E is suddenly applied, the first thing the ion does is to move in the direction of E. Only after picking up a velocity v does the ion feel a Lorentz force $ev \times B$ and begin to move downward. If E is now kept constant, there is no further v_p drift but only a v_E drift. However, if E is reversed, there is again a momentary drift, this time to the left. Thus v_p is a startup drift due to inertia and occurs only in the first half-cycle of each gyration during which E changes. Consequently, v_p goes to zero with ω/Ω.

In other words, the polarization drift can be understood as the change in a particle's guiding center in the direction parallel to $\boldsymbol{E}$ due to the $\boldsymbol{E} \times \boldsymbol{B}$ drift.

To demonstrate this intuition, consider a particle which starts at rest at the origin. Suppose that an electric field is suddenly applied in the $\hat{\boldsymbol{y}}$-direction, such that $\frac{d\boldsymbol{E}}{dt} = \boldsymbol{E}_0\delta(t)$ and $\boldsymbol{E} = \boldsymbol{E}_0 H(t)$ where $H(t)$ is the Heaviside step function. In fig. 5, the polarization drift gives rise to a displacement in the $\hat{\boldsymbol{y}}$-direction. The instantaneous change in the particle's guiding center due to the polarization drift is

$$\Delta \boldsymbol{r}_{gc} = \int \boldsymbol{v}_p \, dt = \frac{m\boldsymbol{E}_0}{qB^2}. \tag{2.31}$$

Because they represent the same physical effect, we would expect the instantaneous change in the particle's guiding center due to the $\boldsymbol{E} \times \boldsymbol{B}$ drift to be identical. Indeed, they are. To see this, transform into a frame moving at velocity $\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}}$; in that frame the particle is simply exhibiting cyclotron motion around a fixed point. Since the particle starts at rest in the laboratory frame, then in the drift frame the perpendicular velocity must be $\boldsymbol{v}_\perp = -\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}}$. Thus the gyro-radius (in the drift frame) becomes

$$\rho = \frac{v_\perp}{\Omega} = \frac{mE_0}{qB^2}.$$

However, the gyro-radius of the particle is also the instantaneous change in guiding center position $\Delta \boldsymbol{r}_{gc}$, since the particle starts at rest at the origin. This is identical to eq. (2.31).

Derivation

Imagine that a charged particle is placed in a spatially constant time-dependent electric field and a static, constant magnetic field. The equation of motion for the particle (eq. (2.1)) becomes

$$\ddot{\boldsymbol{r}}_{gc} + \ddot{\boldsymbol{\rho}} = \frac{q}{m}\big(\boldsymbol{E}(t) + (\dot{\boldsymbol{r}}_{gc} + \dot{\boldsymbol{\rho}}) \times \boldsymbol{B}(\boldsymbol{r})\big). \tag{2.32}$$

Subtracting off the gyro-motion (eq. (2.18)) leaves

$$\ddot{\boldsymbol{r}}_{gc} = \frac{q}{m}\big(\boldsymbol{E}(t) + \dot{\boldsymbol{r}}_{gc} \times \boldsymbol{B}(\boldsymbol{r}_{gc})\big). \tag{2.33}$$

Taking the cross product with $\hat{\boldsymbol{b}}$ and following the same steps as in section 2.1.3 gives

$$\dot{\boldsymbol{r}}_{gc,\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{m}{qB}\ddot{\boldsymbol{r}}_{gc} \times \hat{\boldsymbol{b}}. \tag{2.34}$$

This equation only holds for electric fields which don't vary much over the course of a gyro-orbit, such that $\frac{1}{E\Omega}\frac{\partial \boldsymbol{E}}{\partial t} \ll 1$ and $\frac{\rho}{E}\boldsymbol{\nabla} E \ll 1$. To lowest order,

$$\dot{\boldsymbol{r}}_{gc} = v_{gc\parallel}\hat{\boldsymbol{b}} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}. \tag{2.35}$$

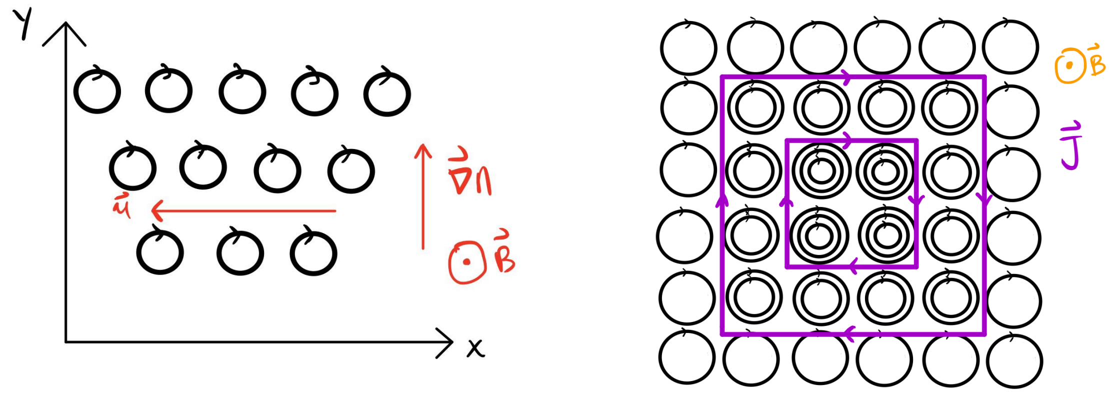
Taking the time derivative gives $\ddot{\boldsymbol{r}}_{gc}$ to first order in ϵ:

$$\ddot{\boldsymbol{r}}_{gc} = \frac{dv_{gc\parallel}}{dt}\hat{\boldsymbol{b}} + \frac{\frac{d\boldsymbol{E}}{dt} \times \boldsymbol{B}}{B^2}. \tag{2.36}$$

Plugging eq. (2.36) into eq. (2.34) gives

$$\dot{\boldsymbol{r}}_{gc\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + \frac{m}{qB^2}\frac{d\boldsymbol{E}_\perp}{dt}. \tag{2.37}$$

The second term is the polarization drift. Note that if we had not assumed that the magnetic field was constant, the curvature and grad-B drifts would have appeared as well.



(a) Magnetization drift (b) Magnetization current

Figure 7: (a) Illustration of the magnetization drift. Magnetized charged particles in a density gradient have an average velocity $\boldsymbol{u}_M$ due to their cyclotron motion in a density gradient. $\boldsymbol{u}_M$ points in the $q\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} n$ direction. The magnetization drift is shown in red. (b) Illustration of the magnetization current. The orbits of charged particles around a magnetic field act as magnetic dipoles and can create a magnetization current in a plasma. The magnetization current is sketched in purple.

2.1.5 Magnetization drift

The magnetization drift is unlike any of the other drifts we've seen so far. While the $\boldsymbol{E} \times \boldsymbol{B}$ drift, the grad-B drift, the curvature drift, and the polarization drift all arise due to single-particle effects, the magnetization drift arises due to collective plasma motion. In particular, gradients in the plasma density, temperature, and/or magnetic field can give rise to an average bulk velocity. This happens even if every individual particle has no average velocity.

Suppose we have a plasma of species σ in a magnetic field $\boldsymbol{B}$ and with density n. Suppose each particle has charge q_σ and mass m_σ, and the plasma has a temperature T_σ. The magnetization drift for species σ is

$$\boldsymbol{u}_{M\sigma} = \frac{-1}{n_\sigma} \boldsymbol{\nabla} \times \Big(\frac{n_\sigma k_B T_{\sigma\perp} \hat{\boldsymbol{b}}}{q_\sigma B}\Big). \tag{2.38}$$

For a plasma with constant temperature in a straight, constant magnetic field, this reduces to

$$\boldsymbol{u}_{M\sigma} = \frac{k_B T_{\sigma\perp}}{n_\sigma q_\sigma B} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} n_\sigma. \tag{2.39}$$

Physical intuition

Figure 7a illustrates the physical origin of this drift. Here positively charged particles orbit clockwise in a magnetic field pointing out of the page. There is a density gradient is in the vertical direction. We're looking at positively charged particles, so they orbit *clockwise* in this region.[8] As we can see, each individual particle simply orbits around the magnetic field, so the perpendicular velocity of each particle's guiding center is 0. However, the *average* velocity $\boldsymbol{u}_M$ is to the left. This is because there are more particles at higher y which move to the left at the bottom of their orbits, and fewer particles at smaller y which move to the right at the top of their orbits.

Magnetization current

The magnetization drift for electrons and ions is in opposite directions. In a quasi-neutral plasma, the magnetization drift creates a current. We can calculate this current as follows:

$$\boldsymbol{J}_M = \sum_\sigma q_\sigma n_\sigma \boldsymbol{u}_{M\sigma} = \boldsymbol{\nabla} \times \Big(\frac{-\sum_\sigma n_\sigma k_B T_\sigma \hat{\boldsymbol{b}}}{B}\Big) = \boldsymbol{\nabla} \times \Big(\frac{-P_\perp \hat{\boldsymbol{b}}}{B}\Big)$$

$$\boldsymbol{J}_M = \boldsymbol{\nabla} \times \boldsymbol{M} \tag{2.40}$$

[8] The direction of orbit of charged particles follows a left-hand rule. For negatively charged particles, the direction of orbit can be found using a right-hand rule.

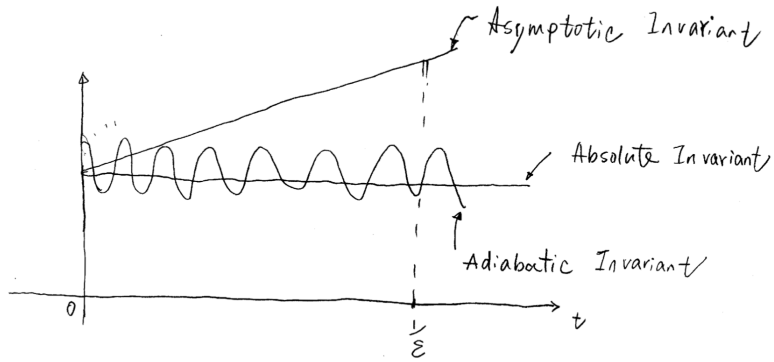


Figure 8: The change in time of an adiabatic invariant, compared with an absolute invariant (such as energy) and an asymptotic invariant.

where $\boldsymbol{M} = -P_\perp \hat{\boldsymbol{b}}/B$.

The magnetization current is, like the magnetization drift, due to the collective effects of a large number of charged particles orbiting around a magnetic field. The physical reason for the magnetization current in a plasma is the same as for the magnetization current in a magnetized material. In a plasma, particles act as magnetic dipoles due to a magnetic field. The curl of the dipole moment gives the magnetization current. This is illustrated schematically in figure 7b, where the purple line represents the magnetization current.

2.1.6 Drift currents

So far, we've derived five drifts, four of which are single-particle guiding-center drifts and one of which is an average bulk drift. Each drift is reproduced below:

$$\boldsymbol{v}_{E\times B} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \quad \boldsymbol{v}_{\boldsymbol{\nabla} B} = \frac{mv_\perp^2}{2}\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla} B}{qB^2}, \quad \boldsymbol{v}_c = \frac{mv_\parallel^2}{qB}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{b}},$$

$$\boldsymbol{v}_p = \frac{m}{qB^2}\frac{d\boldsymbol{E}}{dt}, \quad \boldsymbol{u}_{M\sigma} = \frac{-1}{n_\sigma}\boldsymbol{\nabla}\times\Big(\frac{n_\sigma k_B T_{\sigma\perp}\hat{\boldsymbol{b}}}{q_\sigma B}\Big).$$

With the exception of the $\boldsymbol{E}\times\boldsymbol{B}$ drift, each of these drifts is linear in q_σ. This means that particles of opposite charge will travel in opposite directions, creating a current. Using the definition $\boldsymbol{J} = \sum_\sigma q_\sigma n_\sigma \boldsymbol{u}_\sigma$, we can calculate these currents:

$$\boldsymbol{J}_{E\times B} = 0, \quad \boldsymbol{J}_{\boldsymbol{\nabla} B} = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma \langle v_{\perp,\sigma}^2\rangle}{2q_\sigma}\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla} B}{B^2} = \frac{\boldsymbol{B}\times\boldsymbol{\nabla} B}{B^3}P_\perp,$$

$$\boldsymbol{J}_c = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma\langle v_\parallel^2\rangle}{q_\sigma B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{b}} = \frac{\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{b}}}{B}P_\parallel,$$

$$\boldsymbol{J}_p = \sum_\sigma n_\sigma q_\sigma \frac{m_\sigma}{q_\sigma B^2}\frac{d\boldsymbol{E}}{dt} = (\frac{\rho}{B^2})\frac{d\boldsymbol{E}}{dt}, \quad \boldsymbol{J}_M = \boldsymbol{\nabla}\times(\frac{-P_\perp\hat{\boldsymbol{b}}}{B}).$$

By computing the total perpendicular current $\boldsymbol{J}_\perp = \boldsymbol{J}_{\boldsymbol{\nabla} B} + \boldsymbol{J}_c + \boldsymbol{J}_p + \boldsymbol{J}_M$ and computing $\boldsymbol{J}_{\perp total}\times\boldsymbol{B}$, a tedious calculation gives the perpendicular component of the doubly adiabatic MHD momentum equation. In other words, the single particle guiding center model of a plasma derived here contains the same physical information as the fluid models derived in section 4.

2.2 Adiabatic invariants

Many physical systems have absolute invariants, such as momentum and energy Some physical systems have asymptotic invariants, such as entropy. Hamiltonian systems which depend on slowly varying paramters also have a third class of invariants, called adiabatic invariants. Adiabatic invariants are 'approximately conserved'; they can change over time, but the overall magnitude does not increase or decrease on average so long as changes to the system occur slowly. Figure 8 sketches the difference between an absolute invariant, asymptotic invariant, and an adiabatic invariant.

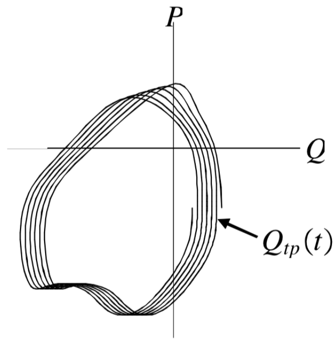


Figure 9: Nearly periodic motion in the $P-Q$ plane. Figure is from [1].

In plasma physics, charged particles moving through electromagnetic fields are a Hamiltonian system and have adiabatic invariants which remain approximately constant, so long as the electromagnetic fields vary slowly in space and time. In particular, a charged particle can have three adiabatic invariants: the first adiabatic invariant $\mu = mv_\perp^2/2B$, the second adiabatic invariant $\mathcal{J} = \oint v_\parallel \, dt$, and a third we don't discuss in these notes. In section 2.2.1 we discuss the general adiabatic invariant for Hamiltonian systems, and in sections 2.2.2 and 2.2.3 we discuss the first and second adiabatic invariants for charged particles in slowly varying electromagnetic fields.

2.2.1 General adiabatic invariant

Suppose that a system with canonical coordinate Q and canonical momentum P evolves according to Hamilton's equations $\dot{Q} = \frac{\partial H}{\partial P}$, and $\dot{P} = -\frac{\partial H}{\partial Q}$. For simplicity, we'll consider the case where Q and P are scalars, rather than vectors. Also suppose that the Hamiltonian H depends on some slowly changing parameter $\lambda(t)$, so that

$$H = H(Q, P, \lambda(t)). \tag{2.41}$$

Suppose also that the canonical coordinates Q and P undergo some nearly periodic motion.[9] In that case, the integral

$$I = \oint P dQ \tag{2.42}$$

is an adiabatic invariant and is approximately conserved as $\lambda(t)$ changes.

How slowly must λ change for I to be conserved? The derivation below depends on $\lambda(t)$ being differentiable from one period to the next. So the integral in eq. (2.42) is constant in the limit that the change in $\lambda(t)$ over any one period is infinitesimal. In the more realistic limit that $\frac{T}{\lambda}\frac{d\lambda}{dt} = \epsilon \ll 1$, where T is the period of the system's periodic motion, then the change in I at any time never becomes greater than some small value, $O(\epsilon)$. Mathematically, this is $\frac{I(t)-I(0)}{I(0)} < O(\epsilon)$ for $0 < t < \frac{1}{\epsilon}$ where $\epsilon \ll 1$.

Derivation

Equation (2.42) was not derived in class. Here I will derive eq. (2.42) relying on the discussion in [1].

We assume that a system (or particle) is undergoing nearly periodic motion in the P-Q plane, as sketched in fig. 9. The integral in eq. (2.42) requires integrating around a nearly periodic orbit; we define the beginning and end of each orbit to be at the turning point Q_{tp} as sketched in fig. 9. Q_{tp} is defined to be the location in the cycle where $\frac{dQ}{dt} = 0$ and Q reaches its maximum value. This definition is arbitrary, but will be mathematically convenient in the derivation below.

Our goal is to show that

$$\frac{dI}{dt} = \frac{d}{dt}\oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} PdQ = 0. \tag{2.43}$$

[9] I don't have a precise mathematical definition of 'nearly periodic'. Intuitively, nearly periodic means that the system returns near its original state after one period.

Now, since $E(t) = H(P, Q, \lambda(t))$, we can in principle invert this to write $P(E(t), Q, \lambda(t))$. So

$$\frac{dI}{dt} = \frac{d}{dt} \oint_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} P(E(t), Q, \lambda(t)) dQ \tag{2.44}$$

To simplify this, we'll use the Leibniz integral rule', which says that

$$\frac{d}{dt} \int_{x=a(t)}^{x=b(t)} f(t,x)\, dx = f(t, b(t)) \frac{db(t)}{dt} - f(t, a(t)) \frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\, dx\,. \tag{2.45}$$

Using the Leibniz integral rule, eq. (2.44) becomes

$$\frac{dI}{dt} = P(E(t+\tau), Q_{tp}(t+\tau), \lambda(t+\tau)) \frac{dQ_{tp}(t+\tau)}{dt} - P(E(t), Q_{tp}(t), \lambda(t)) \frac{dQ_{tp}(t)}{dt} + \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \frac{\partial P}{\partial t} dQ. \tag{2.46}$$

By definition, $\frac{dQ_{tp}(t+\tau)}{dt} = \frac{dQ_{tp}(t)}{dt} = 0$ because those are defined to be the turning points. From the definition of the partial derivative, $\frac{\partial P}{\partial t}$ is defined at constant Q, so we can write it as $(\frac{\partial P}{\partial t})_Q$. The integral is now

$$\frac{dI}{dt} = \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \left(\frac{\partial P}{\partial t}\right)_Q dQ \tag{2.47}$$

Now, let's attack this remaining term.

$$\left(\frac{\partial P}{\partial t}\right)_Q = \left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} \frac{d\lambda}{dt} + \left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} \frac{dE}{dt} \tag{2.48}$$

Now, we can use some tricks to simplify these two terms. Since $E(t) = H(P, Q, \lambda(t))$, then

$$\left(\frac{dH}{dE}\right)_{Q,\lambda} = 1 = \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda} \left(\frac{\partial P}{\partial E}\right)_{Q,\lambda},$$

$$\left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} = \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda}^{-1}.$$

Since E is a function only of time,

$$\left(\frac{dE}{d\lambda}\right)_Q = 0 = \left(\frac{\partial H}{\partial \lambda}\right)_{Q,P} + \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda} \left(\frac{\partial P}{\partial \lambda}\right)_{Q,E},$$

$$\left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} = -\left(\frac{\partial H}{\partial \lambda}\right)_{Q,P} \Big/ \left(\frac{\partial H}{\partial P}\right)_{Q,\lambda}.$$

Plugging these results into eq. (2.48) and then into eq. (2.47), we get

$$\frac{dI}{dt} = \int \frac{1}{(\frac{\partial H}{\partial P})_{Q,\lambda}} \left[\frac{dE}{dt} - \frac{d\lambda}{dt} \left(\frac{\partial H}{\partial \lambda}\right)_{Q,P}\right] dQ \tag{2.49}$$

Using $E(t) = H(P, Q, \lambda(t))$ and Hamilton's equations $\frac{\partial H}{\partial P} = \frac{dQ}{dt}$ and $\frac{\partial H}{\partial Q} = -\frac{dP}{dt}$,

$$\frac{dE}{dt} = \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dt} + \frac{\partial H}{\partial Q} \frac{dQ}{dt} + \frac{\partial H}{\partial P} \frac{dP}{dt} = \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dt}. \tag{2.50}$$

The second and third terms have canceled from Hamilton's equations. Plugging this into eq. (2.49) gives $dI/dt = 0$, which completes our proof of the adiabatic invariance of I.

This proof is not rigorous, as it depends on $d\lambda/dt$ being infinitesimally small. If $d\lambda/dt$ is greater than zero, then τ depends on t as well, and the first term in the Leibniz integral rule becomes $\partial Q_{tp}(t+\tau)/\partial t + \partial Q_{tp}(t+\tau)/\partial\tau \, \partial\tau/\partial\lambda \, d\lambda/dt$. Professor Qin said that if students are interested in seeing a rigorous proof of adiabatic invariance, he can point them to one.

2.2.2 First adiabatic invariant

The magnetic moment of a charged particle in a plasma is

$$\mu = \frac{mv_\perp^2}{2B} = \frac{KE_\perp}{B}. \tag{2.51}$$

μ is the first adiabatic invariant for individual charged particles in electromagnetic fields. The first adiabatic invariant arises due to the nearly periodic motion of particles gyrating around magnetic field lines.

Physical intuition

From undergraduate physics, the magnetic moment of a loop carrying current I with area A is IA. We can think of charged particles in a magnetic field as being little loops of current carrying material. The area of that loop is $\pi\rho^2 = \pi\frac{v_\perp^2}{\Omega^2}$ and the current is $\frac{q}{T} = \frac{q\Omega}{2\pi}$. Putting this together, and using $\Omega = \frac{qB}{m}$ as usual, gives

$$IA = \frac{\pi v_\perp^2}{\Omega^2}\frac{q\Omega}{2\pi} = \frac{mv_\perp^2}{2B} = \mu.$$

This implies that the magnetic moment of charged particles in a plasma has the same physical meaning as the magnetic moment of a magnetic dipole.

As we know from undergraduate physics, the force on a magnetic dipole with magnetic moment μ in a changing magnetic field is

$$F_\parallel = -\mu\boldsymbol{\nabla}_\parallel B.$$

This explains why the magnetic moment of charged particles is conserved: as particles move to regions of stronger field, they feel a force slowing them down in the parallel direction, and due to conservation of energy their velocity in the perpendicular direction needs to increase. This intuition is discussed further in section 2.3.

Derivation

Imagine there is a particle of mass m and charge q in a magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ which changes slowly in space and time. If this particle does not collide with other particles, then energy is conserved:

$$\frac{d}{dt}(mv_\perp^2/2 + mv_\parallel^2/2) = \frac{d}{dt}(\mu B + \frac{1}{2}mv_\parallel^2) = 0, \tag{2.52}$$

$$\frac{d\mu}{dt}B + \mu\frac{dB}{dt} + mv_\parallel\frac{dv_\parallel}{dt} = 0. \tag{2.53}$$

From section 2.1.3 we know that $\frac{dB}{dt} = \frac{\partial s}{\partial t}\frac{\partial B}{\partial s} = v_\parallel(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})B$. Next we take the dot product of $\hat{\boldsymbol{b}}$ with eq. (2.21) to get

$$\hat{\boldsymbol{b}}\cdot(\langle\ddot{\boldsymbol{r}}_{gc}\rangle) = \frac{-\mu(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla})B(\boldsymbol{r}_{gc})}{m}. \tag{2.54}$$

Note that eq. (2.21) was derived assuming that $\boldsymbol{B}(\boldsymbol{r})$ varies slowly in space. Note also that $\hat{\boldsymbol{b}}\cdot(\langle\ddot{\boldsymbol{r}}_{gc}\rangle) = \langle\frac{dv_\parallel}{dt}\rangle$. Plugging these results into eq. (2.53), averaging over a gyro-period, and assuming that $v_\parallel$ is approximately constant over a gyro-period, we have that

$$0 = \frac{d\mu}{dt}B + \mu v_\parallel\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}B - mv_\parallel\frac{\mu\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}B}{m} = \frac{d\mu}{dt}B \tag{2.55}$$

$$\frac{d\mu}{dt}B = 0. \tag{2.56}$$

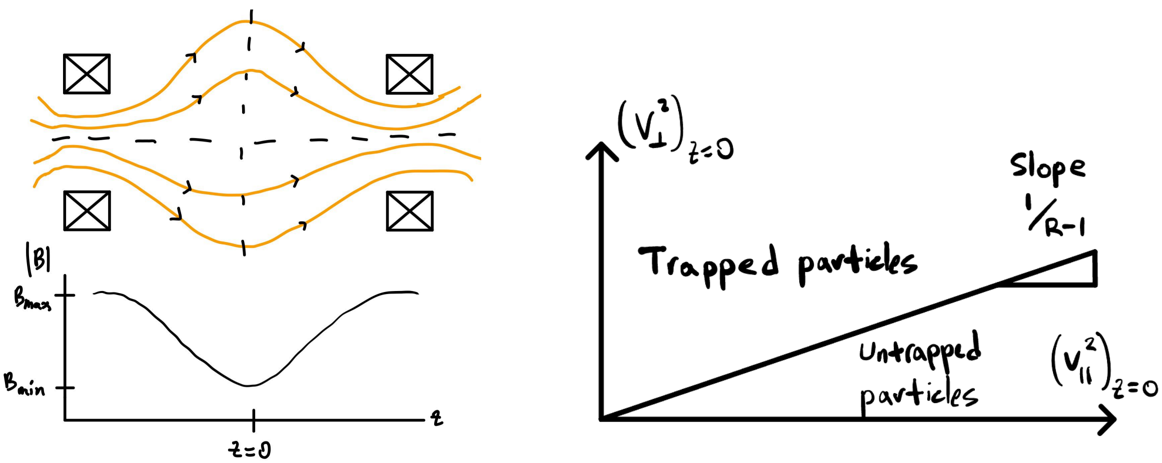
Assuming that $\boldsymbol{B}$ varies slowly in space, μ is approximately conserved for charged particles orbiting in electromagnetic fields. It turns out that μ is conserved for fields that vary slowly in time, as well.

2.2.3 Second adiabatic invariant

Suppose a particle of charge q is moving through arbitrary electromagnetic fields. Suppose that particle's guiding center undergoes some approximately periodic motion. For example, the particle might be bouncing back and forth between two regions of space. Also suppose that the electromagnetic fields are changing slowly, such that $\frac{\tau_{bounce}}{B}\frac{d\boldsymbol{B}}{dt} \ll 1$ meaning the timescale over which the electromagnetic fields change is much longer than the particle's bound period τ_B. If these conditions are met, then the quantity $\mathcal{J}$ is an adiabatic invariant for each particle:

$$\mathcal{J} = \oint v_\parallel dt \tag{2.57}$$

$\mathcal{J}$ is called the second adiabatic invariant. Conservation of the second adiabatic invariant was not derived in class.



(a) (b)

Figure 10: (a) Illustration of the magnetic mirror (top) and the magnetic field magnitude as a function of z (bottom). The magnetic field is cylindrically symmetric. The magnetic field lines are shown in orange. (b) Plot of eq. (2.60) showing the particles which are trapped and which aren't. As we can see, particles with larger perpendicular velocities are trapped.

2.3 Mirror machine

One of the most basic and important concepts in single particle motion is the mirror machine. The simplest mirror machine consists of two cylindrically symmetric current-carrying coils which create an axisymmetric magnetic field. This is illustrated in Figure 10a. More complex versions of the magnetic mirror might involve electric fields, time-varying magnetic fields, etc.

Trapping condition

Some particles will be trapped inside a magnetic mirror and others will escape through the ends of the mirror in fig. 10a. We'd like to derive the condition for which particles are trapped and which are not.

To do so, we'll use two invariants: conservation of energy $E = mv_\perp^2/2 + mv_\parallel^2/2 + q\phi$, and the first adiabatic invariant $\mu = mv_\perp^2/2B$.[10] The strategy is to equate the invariants E and μ at the midplane ($z = 0$) of the magnetic mirror, where the magnetic field strength is lowest, to the value of z with the maximum possible value of the magnetic field. The particles will be trapped if the maximum possible value of z occurs at or before the maximum in the magnetic field B_{max}. They will escape if $v_\parallel^2 > 0$ at this value of z.

Assuming the electric potential $\phi = 0$, then setting the invariants equal gives

$$E = \big(\frac{1}{2}mv_\perp^2\big)_{z=0} + \big(\frac{1}{2}mv_\parallel^2\big)_{z=0} = \big(\frac{1}{2}mv_\perp^2\big)_{B=B_{max}} \tag{2.58}$$

and

$$\mu = \frac{1}{B_{min}}\big(\frac{1}{2}mv_\perp^2\big)_{z=0} = \frac{1}{B_{max}}\big(\frac{1}{2}mv_\perp^2\big)_{B=B_{max}}. \tag{2.59}$$

A simple calculation – which you should do on your own, because you will be asked to do it multiple times over this course – reveals that the trapped particles satisfy

$$\left(\frac{v_\perp^2}{v_\parallel^2}\right)_{z=0} \geq \frac{1}{R-1} \tag{2.60}$$

where $R = B_{max}/B_{min}$.

[10] In more advanced calculations, such as when the fields in the mirror machine are changing slowly relative to the bounce time of particles between the ends of the mirror, the second adiabatic invariant is used as well.

Physical intuition

The trapping equation (eq. (2.60)) has a straightforward interpretation: particles with high perpendicular velocities are trapped in the mirror, while particles with high parallel velocities are lost from the mirror. This is because, as a particle travels into a region with higher magnetic field strength, due to conservation of μ $v_\perp^2$ must increase as well. Due to conservation of energy, this decreases $v_\parallel^2$. If the particle has sufficiently high $v_\parallel^2$ at $z = 0$, then $v_\perp^2$ will increase as $|B|$ increases, but not by enough to bring the parallel velocity to zero at B_{max}. Those particles will be lost from the mirror.

If the mirror ratio $R \to 1$, then the ratio required for trapping $v_\perp^2/v_\parallel^2 \to \infty$, and so no particles will be trapped. This makes sense, because if $R = 1$ then the magnetic field is constant and no particles are trapped in a constant magnetic field.

The Lorentz force is the mechanism by which this trapping occurs. To illustrate how this happens, consider the magnetic mirror machine of figure 10a. The magnetic field is given by $\boldsymbol{B} = B_r\hat{\boldsymbol{r}} + B_z\hat{\boldsymbol{z}}$ in a cylindrical coordinate system. Imagine a particle of charge $+q$ starts at $z = 0$ and orbits the $r = 0$ axis towards positive-z, where the magnetic field increases in strength. Since $q > 0$, $v_\perp$ will be in the negative-θ direction. Since the magnetic field lines are converging towards $r = 0$ inside the mirror, then $B_r = -|B_r|$. The Lorentz force $\boldsymbol{v} \times \boldsymbol{B}$ will have two components. First, a force $\boldsymbol{v}_\perp \times B_r\hat{\boldsymbol{r}} = -\hat{\boldsymbol{\theta}}|v_\perp| \times -|B_r|\hat{\boldsymbol{r}} \propto -\hat{\boldsymbol{z}}$, accelerating the particle back towards the midplane at $z = 0$. Second, a force $v_z\hat{\boldsymbol{z}} \times \boldsymbol{B}_r$ in negative-$\hat{\boldsymbol{\theta}}$ direction, increasing the perpendicular velocity of the particle. As a charged particle gyro-orbits towards positive-z, the Lorentz force causes $v_\parallel^2$ to decrease and $v_\perp^2$ to increase. As we've seen, μ-conservation arises due to the Lorentz force.

2.4 Magnetic surfaces

A magnetic surface (or flux surface) is a surface in space where all of the magnetic field lines on the surface stay on the surface. Mathematically, this implies that $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0$ everywhere on the surface, where $\hat{\boldsymbol{n}}$ is the normal vector perpendicular to the surface.

Magnetic field lines are not in any way guaranteed to form magnetic surfaces. The most general behavior of magnetic field lines is stochastic (i.e. random) behavior, meaning a given magnetic field line, if followed forever, will fill a volume in space. Magnetic field lines also do not necessarily close in on themselves, even though $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ and even in the special case where we have magnetic surfaces. A magnetic field line on a magnetic surface might go around the surface forever, never closing on itself but filling the surface. Magnetic field lines can close on themselves after some finite distance, but in general they do not.

Axisymmetric magnetic fields

In axisymmetric magnetic fields (where $\partial/\partial\phi \to 0$ in a cylindrical coordinate system), surfaces of constant rA_ϕ are magnetic surfaces. To prove this, we want to show that $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0$ everywhere on the surface of constant rA_ϕ. Since $\hat{\boldsymbol{n}} = \boldsymbol{\nabla}(rA_\phi)$, then we want to show that

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}(rA_\phi) = 0. \tag{2.61}$$

Using $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{A} = A_r(r,z)\hat{\boldsymbol{r}} + A_\phi(r,z)\hat{\boldsymbol{\phi}} + A_z(r,z)\hat{\boldsymbol{z}}$ due to axisymmetry,

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\hat{\boldsymbol{\phi}} - \frac{\partial A_\phi}{\partial z}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\hat{\boldsymbol{z}}. \tag{2.62}$$

Computing the dot product gives

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}(rA_\phi) = -\frac{\partial A_\phi}{\partial z}\frac{\partial}{\partial r}(rA_\phi) + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\frac{\partial (rA_\phi)}{\partial z} = 0. \tag{2.63}$$

3 Kinetic theory

It is only the plasma itself which does not understand how beautiful the theories are and absolutely refuses to obey them.

HANNES ALFVÉN

Consider the field of classical fluid mechanics. As with all states of matter, fluids are made up of individual molecules or atoms. Perhaps the most fundamental assumption in fluid mechanics is the *continuum assumption*. Under this assumption, all quantities are treated as continuous and well-defined at each point in space. Strictly speaking, this requires that for each quantity and each point in space we set the value of that quantity equal to the average value of that quantity over a volume large enough to contain many molecules but much smaller than the relevant macroscopic lengths of the fluid. In an ideal fluid, there is a well-defined, smooth mass distribution $\rho(\boldsymbol{r}, t)$ and mean velocity $\boldsymbol{u}(\boldsymbol{r}, t)$ at each point in space $\boldsymbol{r}$.

Even in classical fluid mechanics, the continuum assumption is not always valid. Wikipedia writes:

> Those problems for which the continuum hypothesis fails, can be solved using statistical mechanics. To determine whether or not the continuum hypothesis applies, the Knudsen number, defined as the ratio of the molecular mean free path to the characteristic length scale, is evaluated. Problems with Knudsen numbers below 0.1 [are typically well-approximated] using the continuum hypothesis, but the molecular approach (statistical mechanics) can be applied for all ranges of Knudsen numbers.

As we know from section 1, in a plasma the mean free path (i.e., the average distance between collisions) is significantly longer than the Debye length, the characteristic scale length over which a plasma remains quasi-neutral. Actually, the mean free path is often much longer than any of the relevant scale lengths. Consider, for example, a fusion-relevant plasma with number density $n \approx 10^{20}\mathrm{m}^{-3}$ and temperature 1 KeV. In this plasma, the mean free path would be roughly $3\mathrm{km}$, much longer than any of the relevant scale lengths. Thus, the Knudsen number of a plasma is often very large, and the continuum assumptions underlying fluid models may not be valid. This motivates us to develop a molecular approach which is called kinetic theory.

Section overview

In the previous section, we considered the motion of single charged particles evolving under prescribed electric and magnetic fields. In this section, we consider the behavior of a large number of charged particles interacting with electromagnetic fields.

We begin by examining the time-evolution of particles in 6-D phase space. The 6-D phase space has 3 spatial dimensions $\boldsymbol{r}$ and 3 velocity dimensions $\boldsymbol{v}$. We let a function $N(\boldsymbol{r}(t), \boldsymbol{v}(t))$ define the positions and velocities of particles in phase space, and derive an equation for the time-evolution of N called the Klimontovich equation. Combined with the Lorentz force law and Maxwell's equations, this set of equations is exactly equivalent to a number of charged particles interacting through electromagnetic forces.

We then use a technique called ensemble-averaging to go from $N(\boldsymbol{r}(t), \boldsymbol{v}(t))$, which tracks individual particles, to $f(\boldsymbol{r}, \boldsymbol{v}, t)$, a smooth distribution function which tracks the density of particles in phase space. See fig. 12b for an illustration of ensemble-averaging. The equation for the time-evolution of f is called the Vlasov equation (in some fields it is called the Boltzmann equation). The Vlasov equation is the most important equation in kinetic theory; it is given by

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\boldsymbol{v} f) + \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (\boldsymbol{a} f) = C(f) \tag{3.1}$$

where the collision operator $C(f)$ captures the effects of collisions between particles.

While fluid models typically assume that the fluid (or plasma) is in a maximum-entropy state called a Maxwellian at each point in space, the Vlasov equation allows the distribution function to evolve due to three effects: free-streaming, forces, and collisions. The Vlasov equation can be interpreted as a way of maintaining the continuum assumption while allowing for low collisionality. Because fluid models replace the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ with a density $n(\boldsymbol{r}, t)$, three components of the mean velocity $\boldsymbol{u}(\boldsymbol{r}, t)$, and a six-dimensional antisymmetric pressure tensor $P_{ij}(\boldsymbol{r}, t)$, they go from an infinite-dimensional distribution function to a 10-dimensional representation of the distribution function, and thus remove potentially important information about the plasma. As a result, using a fluid model limits the range of plasma systems that we can study.

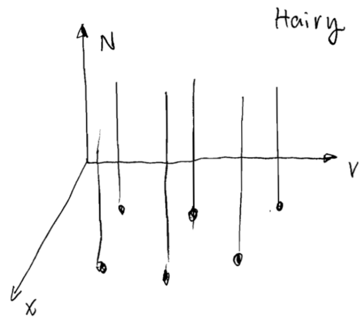


Figure 11: Visualization of N. Each delta function represents the trajectory of a particle in phase space.

We end this section by examining some of the properties of the Vlasov equation, and some examples of collision operators.

3.1 Klimontovich equation

Suppose we have N_0 particles in some region of space. The function $N(\boldsymbol{r}(t), \boldsymbol{v}(t))$ describes the evolution of those N_0 particles in phase space:

$$N(\boldsymbol{r}(t), \boldsymbol{v}(t)) = \sum_{i=1}^{N_0} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))\delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)). \tag{3.2}$$

$\boldsymbol{r}_i(t)$ and $\boldsymbol{v}_i(t)$ represent the position and velocity of the ith particle. Note that the units of N are $L^{-3}V^{-3}$. $N(\boldsymbol{r}, \boldsymbol{v}, t)$ is sketched in fig. 11.

The time-evolution of N is given by the Klimontovich equation

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla_r} \cdot (\boldsymbol{v} N) + \boldsymbol{\nabla_v} \cdot (\boldsymbol{a} N) = 0. \tag{3.3}$$

For a fully-ionized plasma, the Klimontovich equation becomes

$$\frac{\partial N_\sigma}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla_r} N_\sigma + \frac{q_\sigma}{m_\sigma}(\boldsymbol{E}_m + \boldsymbol{v} \times \boldsymbol{B}_m) \cdot \boldsymbol{\nabla_v} N_\sigma = 0 \tag{3.4}$$

where $\boldsymbol{E}_m$ and $\boldsymbol{B}_m$ represent the electric and magnetic fields. The subscript m refers to 'microscopic' and makes clear that, in the microscopic description of a plasma in the Klimontovich equation, the electric and magnetic fields have microscopic fluctuations due to the presence of discrete charged particles.

Derivation

To compute the time-evolution of N, we take the partial derivative with respect to time and use the chain rule:

$$\frac{\partial N(\boldsymbol{r}(t), \boldsymbol{v}(t))}{\partial t} = \sum_{i=1}^{N_0} \frac{\partial N}{\partial \boldsymbol{r}_i} \cdot \frac{d\boldsymbol{r}_i}{dt} + \frac{\partial N}{\partial \boldsymbol{v}_i} \cdot \frac{d\boldsymbol{v}_i}{dt}. \tag{3.5}$$

Using eq. (3.2), we have that

$$\frac{\partial N}{\partial \boldsymbol{r}_i} = \frac{\partial \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))}{\partial \boldsymbol{r}_i}\delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)) = -\frac{\partial \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))}{\partial \boldsymbol{r}}\delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)) \tag{3.6}$$

and similarly

$$\frac{\partial N}{\partial \boldsymbol{v}_i} = -\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))\frac{\partial \delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t))}{\partial \boldsymbol{v}}. \tag{3.7}$$

In eq. (3.5), we can replace $\frac{d\boldsymbol{r}_i}{dt}$ with $\boldsymbol{v}_i$, and $\frac{d\boldsymbol{v}_i}{dt}$ with $\boldsymbol{a}_i$. Plugging 3.6 and 3.7 into 3.5 gives

$$\frac{\partial N}{\partial t} = -\sum_{i=1}^{N_0} \boldsymbol{v_i} \cdot \frac{\partial \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))}{\partial \boldsymbol{r}}\delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)) + \boldsymbol{a}_i \cdot \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t))\frac{\partial \delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t))}{\partial \boldsymbol{v}}. \tag{3.8}$$

We then pull the gradients outside of each term. There is no problem in the first term, because $\boldsymbol{v}$ and $\boldsymbol{r}$ are independent variables and so $\boldsymbol{v}_i$ doesn't depend on $\boldsymbol{r}$. I'm not convinced that this is legal to do for the second term, because $\boldsymbol{a}_i$ might depend on $\boldsymbol{v}_i$. Nevertheless, this is what was done in class. Assuming that we can pull the gradient out of the second term, we have

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial \boldsymbol{r}} \cdot \sum_{i=1}^{N_0} \Big(\boldsymbol{v_i} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t)) \delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)) \Big) - \frac{\partial}{\partial \boldsymbol{v}} \cdot \sum_{i=1}^{N_0} \Big(\boldsymbol{a}_i \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_i(t)) \delta^{(3)}(\boldsymbol{v} - \boldsymbol{v}_i(t)) \Big). \tag{3.9}$$

Next, we can simplify $\boldsymbol{v}_i$ and $\boldsymbol{a}_i$. Because of the delta functions, $\boldsymbol{v}_i(t)$ becomes $\boldsymbol{v}$ and $\boldsymbol{a}_i(t)$ becomes $\boldsymbol{a}$.[11] Pulling these out of the summation, we can replace the sum with N and are left with the Klimontovich equation, eq. (3.3).

Since $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables, $\boldsymbol{\nabla_r} \cdot (\boldsymbol{v} N) = \boldsymbol{v} \cdot \boldsymbol{\nabla_r} N$. If $\boldsymbol{\nabla_v} \cdot \boldsymbol{a} = 0$, then we can rewrite the Klimontovich equation as

$$\frac{\partial N}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla_r} N + \boldsymbol{a} \cdot \boldsymbol{\nabla_v} N = 0. \tag{3.10}$$

Klimontovich-Maxwell system

In plasma physics, the most important example of $\boldsymbol{a}$ is the Lorentz force

$$\boldsymbol{a} = \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{3.11}$$

For the Lorentz force, $\boldsymbol{\nabla_v a} = 0$, which we can prove as follows. $\boldsymbol{\nabla_v} = \frac{\partial}{\partial v_x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial v_y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial v_z}\hat{\boldsymbol{z}}$, so $\boldsymbol{\nabla_v} \cdot \boldsymbol{a} = \frac{\partial a_x}{\partial v_x} + \frac{\partial a_y}{\partial v_y} + \frac{\partial a_z}{\partial v_z}$. Each component $\frac{\partial a_i}{\partial v_i} = \frac{q}{m}(\frac{\partial E_i}{\partial v_i} + \frac{\partial}{\partial v_i}(\boldsymbol{v} \times \boldsymbol{B})_i)$ equals 0 because the ith component of $\boldsymbol{v} \times \boldsymbol{B}$ includes the other two components of $\boldsymbol{v}$ but does not include v_i. Thus, eq. (3.10) can be used for the Lorentz force. For a fully-ionized plasma where each species of plasma particles σ has charge q and mass m, the acceleration is $\boldsymbol{a} = \frac{q_\sigma}{m_\sigma}(\boldsymbol{E}_m + \boldsymbol{v} \times \boldsymbol{B}_m)$. We ignore gravitational acceleration. $\boldsymbol{E}_m$ and $\boldsymbol{B}_m$ are the magnetic fields, which contain microscopic fluctuations due to the discrete particles in the Klimontovich representation. For a fully-ionized plasma, the Klimontovich equation is therefore given by eq. (3.4).

Solving eq. (3.4) requires not only initial and boundary conditions on N_σ, but also Maxwell's equations describing the time-evolution of $\boldsymbol{E}_m$ and $\boldsymbol{B}_m$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_m = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int N_\sigma(\boldsymbol{r}(t), \boldsymbol{v}(t))\, d^3\boldsymbol{v} \tag{3.12}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_m = 0 \tag{3.13}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}_m = -\frac{d\boldsymbol{B}_m}{dt} \tag{3.14}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_m = \mu_0 \sum_\sigma q_\sigma \int \boldsymbol{v} N_\sigma(\boldsymbol{r}(t), \boldsymbol{v}(t))\, d^3\boldsymbol{v} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}_m}{\partial t}. \tag{3.15}$$

This system of equations, along with eq. (3.4), is called the Klimontovich-Maxwell equations. They are equivalent to a number of charged particles interacting through electromagnetic forces.

Physical intuition

The Klimontovich equation eq. (3.3) is a continuity equation for particles in phase space. Geometrically, the Klimontovich equation states that the number of particles leaving a region in phase space is equal to the number of particles traveling across the boundary of that region in phase space. To show this, we integrate the Klimontovich equation over a volume V in 6-D phase space, where $\boldsymbol{z} = \{\boldsymbol{r}, \boldsymbol{v}\}$ represents the position in phase space and $\boldsymbol{v_z} = \{\boldsymbol{v}, \boldsymbol{a}\}$ represents the velocity in phase space. This gives

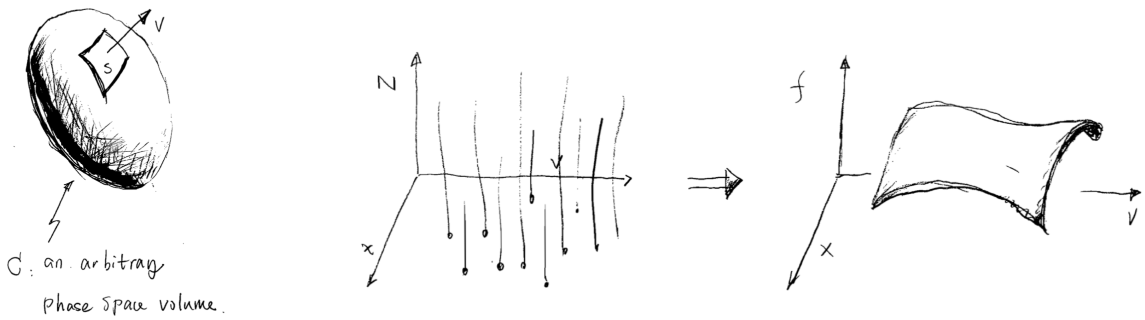
$$\frac{\partial}{\partial t}\left[\int_V N(\boldsymbol{z}(t))\, d^6\boldsymbol{z}\right] = -\int_V \boldsymbol{\nabla_z} \cdot (N \boldsymbol{v_z})\, d^6\boldsymbol{z}\,. \tag{3.16}$$

Using the divergence theorem,

$$\frac{\partial}{\partial t}\left[\int_V N(\boldsymbol{z}(t))\, d^6\boldsymbol{z}\right] = -\int_{\partial V} N \boldsymbol{v_z} \cdot d\boldsymbol{A}. \tag{3.17}$$

Figure 12a illustrates one such region in phase space.

[11] Why can't we apply the delta function to eq. (3.8), pull the dot products out of the equation, and then pull the gradients out of the sum to get $\frac{\partial N}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla_r} N + \boldsymbol{a} \cdot (\boldsymbol{\nabla_v} N) = 0$? Because the derivative of a delta function is not a delta function. We have to first move the derivative off the delta function before simplifying.



(a) (b)

Figure 12: (a) 3-D Visualization of the 6-D volume V (labeled C) in phase space which we consider. (b) Visualization of ensemble-averaging N to get f.

3.2 Vlasov equation

We'd like to find an equation for the evolution of a smooth function f which captures the density of particles in phase space, thereby replacing a discrete model with a continuum model. This smoothing process, which we call ensemble averaging, is sketched in fig. 12b. The result is the Vlasov equation, reproduced below:

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\boldsymbol{v} f) + \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (\boldsymbol{a} f) = C(f). \tag{3.18}$$

For fully ionized plasmas, the Vlasov equation for particle species σ is

$$\frac{\partial f_\sigma}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} f_\sigma + \frac{q_\sigma}{m_\sigma}\big(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\big) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma = C(f_\sigma). \tag{3.19}$$

Derivation

We derive eq. (3.19); deriving eq. (3.18) follows the same process.

An ensemble is defined as all of the possible microstates corresponding to a given macrostate. To derive the Vlasov equation, we'll average the Klimontovich-Maxwell system (eqs. (3.4) and (3.12) to (3.15)) over every possible microstate consistent with the macrostate. We represent this ensemble averaging with brackets, $\langle \dots \rangle$. The smooth distribution function $f_\sigma(\boldsymbol{r}, \boldsymbol{v}, t) \equiv \langle N_\sigma(\boldsymbol{r}(t), \boldsymbol{v}(t)) \rangle$, while the ensemble-averaged (smooth) electric and magnetic fields $\boldsymbol{B} \equiv \langle \boldsymbol{B}_m \rangle$ and $\boldsymbol{E} \equiv \langle \boldsymbol{E}_m \rangle$.[12] Let us also define $\delta N_\sigma \equiv N_\sigma - f_\sigma$, $\delta \boldsymbol{E} \equiv \boldsymbol{E}_m - \boldsymbol{E}$, and $\delta \boldsymbol{B} \equiv \boldsymbol{B}_m - \boldsymbol{B}$.

We first ensemble-average Maxwell's equations. Using $N_\sigma = f_\sigma + \delta N_\sigma$, $\boldsymbol{B}_m = \boldsymbol{B} + \delta \boldsymbol{B}$, $\boldsymbol{E}_m = \boldsymbol{E} + \delta \boldsymbol{E}$, and $\langle \delta N_\sigma \rangle = \langle \delta \boldsymbol{B} \rangle = \langle \boldsymbol{E} \rangle = 0$, Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} \sum_\sigma q_\sigma \int f_\sigma(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3\boldsymbol{v} \tag{3.20}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.21}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3.22}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \sum_\sigma q_\sigma \int \boldsymbol{v} f_\sigma(\boldsymbol{r}, \boldsymbol{v}, t)\, d^3\boldsymbol{v} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{3.23}$$

These are the same as Maxwell's equations before ensemble averaging, just with m removed. The $\delta \boldsymbol{B}$ and δE terms are removed in the ensemble-averaging process because the equations are linear.

Second, we ensemble-average eq. (3.4), the Klimontovich equation with the Lorentz force. Using $\langle \delta N_\sigma \rangle = 0$, this gives

$$\frac{\partial f_\sigma}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} f_\sigma + \left\langle \frac{q_\sigma}{m_\sigma}((\boldsymbol{E} + \delta \boldsymbol{E}) + \boldsymbol{v} \times (\boldsymbol{B} + \delta \boldsymbol{E})) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} (f_\sigma + \delta N_\sigma) \right\rangle = 0.$$

[12]The notation $\equiv$ means 'is defined to be'.

Setting the terms first-order in $\delta\boldsymbol{E}$, $\delta\boldsymbol{B}$, and δN_σ equal to zero, this gives

$$\frac{\partial f_\sigma}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma = -\left\langle \frac{q_\sigma}{m_\sigma}\big(\delta\boldsymbol{E}+\boldsymbol{v}\times\delta\boldsymbol{B}\big)\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\delta N_\sigma\right\rangle \tag{3.24}$$

The correlation term on the right hand side accounts for the effect of particle-particle interactions, i.e. collisions. For simplicity, we can replace the correlation term with a collision operator $C(f_\sigma)$, which derives the eq. (3.19).

Physical intuition

Suppose that $\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\boldsymbol{a}=0$ and there are no collisions, so $C(f)=0$. eq. (3.18) can then be rewritten as

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}} f + \boldsymbol{a}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f = 0. \tag{3.25}$$

In phase space, eq. (3.25) can be rewritten as

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_{\boldsymbol{z}}\cdot\boldsymbol{\nabla}_{\boldsymbol{z}} f = 0 \tag{3.26}$$

where $\boldsymbol{v}_{\boldsymbol{z}} = \{\boldsymbol{v},\boldsymbol{a}\}$ is the velocity in phase space. Equation (3.26) is a convective derivative in phase space,

$$\frac{df}{dt} = 0. \tag{3.27}$$

This implies that as f advects in phase space with velocity $\boldsymbol{v}_{\boldsymbol{z}}$, the magnitude of f is unchanged. This is an identical result to Liouville's theorem for Hamiltonian systems. If collisions are included, then the magnitude of f can change as f is advected through phase space.

At first glance, it isn't clear what the difference between eq. (3.10) and eq. (3.25) are. Both assume that $\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\boldsymbol{a}=0$, and they algebraically are the same equation: one is $\frac{dN}{dt}=0$, the other $\frac{df}{dt}=0$. Nevertheless, the Klimontovich equation (eq. (3.10) and the collisionless Vlasov equation (eq. (3.25)) represent two different physical situations. The Klimontovich equation describes the evolution of a *single* microstate, and includes collisions between particles. The collisionless Vlasov equation, by contrast, describes the ensemble-averaged macrostate, and ignores the effects of collisions between particles.

3.2.1 Moments of the distribution function

By computing moments of the distribution function f, we can compute macroscopic quantities of interest. The particle number density is

$$n_\sigma(\boldsymbol{r},t) = \int f_\sigma(\boldsymbol{r},\boldsymbol{v},t)\, d^3\boldsymbol{v}\,. \tag{3.28}$$

The number density is the zeroth moment of f. The total number of particles of species σ in the plasma

$$N_\sigma = \int f_\sigma(\boldsymbol{r},\boldsymbol{v},t)\, d^3\boldsymbol{r}\, d^3\boldsymbol{v}\,. \tag{3.29}$$

The mean velocity of species σ

$$\boldsymbol{u}_\sigma = \frac{1}{n_\sigma}\int \boldsymbol{v} f_\sigma(\boldsymbol{r},\boldsymbol{v},t)\, d^3\boldsymbol{v}\,. \tag{3.30}$$

The mean velocity is the first moment of f with respect to velocity, divided by the density. In words, the mean velocity $\boldsymbol{u}$ is the first moment of f with respect to velocity, divided by the density.[13] The plasma energy per volume in the particle kinetic energy for species σ

$$E = \int \frac{1}{2} m_\sigma(\boldsymbol{v})^2 f_\sigma\, d^3\boldsymbol{v}\,. \tag{3.31}$$

E accounts for both the thermal energy of the plasma as well as the kinetic energy of the mean plasma flow.

3.2.2 Properties of collisionless Vlasov-Maxwell equations

We consider the Vlasov-Maxwell system of equations eqs. (3.19) to (3.23), without collisions so that $C(f_\sigma)=0$.

[13] Throughout these notes, I use $\boldsymbol{u}$ to represent a mean fluid velocity, and $\boldsymbol{v}$ or v to represent a single-particle velocity or thermal speed.

Conservation of particles

The total number of particles is given by eq. (3.29). Taking the derivative with respect to time, this becomes

$$\frac{d}{dt}\int f_\sigma(\boldsymbol{r},\boldsymbol{v},t)\,d^3\boldsymbol{r}\,d^3\boldsymbol{v} = \int \frac{\partial f_\sigma}{\partial t}\,d^3\boldsymbol{r}\,d^3\boldsymbol{v} = -\int \boldsymbol{\nabla_z}\cdot(\boldsymbol{v_z} f)\,d^6\boldsymbol{z}\,. \tag{3.32}$$

Integrating by parts, this last term equals zero at the domain boundaries.

Conservation of energy

The total energy (plasma plus electromagnetic) is conserved through the evolution of f under the Vlasov-Maxwell equation. The total energy

$$\mathcal{E} = \frac{1}{2}\int d^3\boldsymbol{r}\left[\epsilon_0 E^2 + \frac{B^2}{\mu_0} + \sum_\sigma \int d^3\boldsymbol{v} m_\sigma v^2 f_\sigma\right] \tag{3.33}$$

is constant in time. This is proven in a homework problem.

Conservation of momentum

The total momentum is also conserved:

$$\boldsymbol{\mathcal{P}} = \int d^3\boldsymbol{r}\left[\epsilon_0 \boldsymbol{E}\times\boldsymbol{B} + \sum_\sigma \int d^3\boldsymbol{v} m_\sigma \boldsymbol{v} f_\sigma\right] \tag{3.34}$$

This is also proven in a homework problem.

Constants of motion

If $c_i(\boldsymbol{r},\boldsymbol{v},t,\boldsymbol{B}(\boldsymbol{r},t),\boldsymbol{E}(\boldsymbol{r},t))$ is a constant of motion for a single particle, such that

$$\frac{dc_i}{dt} = \frac{\partial c_i}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} c_i + \frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla_v} c_i = 0 \tag{3.35}$$

then any function $f(c_1, c_2, ...,)$ which is a function of c_i's is a solution of the collisionless Vlasov equation. This is easily shown as

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial c_i}\frac{dc_i}{dt} = 0. \tag{3.36}$$

3.2.3 Entropy of a distribution function

The entropy of a distribution function f is

$$S = -\int f \ln f\, d^3\boldsymbol{v}\, d^3\boldsymbol{r}\,. \tag{3.37}$$

The maximum-entropy distribution function subject to the constraint that particles, momentum, and energy are conserved is a Maxwellian.

Derivation

From statistical mechanics, the entropy is

$$S = k_B \ln C \tag{3.38}$$

where k_B is the Boltzmann constant and C is the number of microscopic states corresponding to the given macrostate. To calculate the entropy of a distribution function, we need to calculate C, the number of microstates consistent with the macrostate defined via $f(\boldsymbol{r},\boldsymbol{v},t)$.

Imagine we have N distinguishable pegs we can put into N holes, so that exactly 1 peg goes into each hole. Since the pegs are distinguishable, we have $N!$ ways of ordering the N pegs into the N holes (N options for the first peg, $N-1$ options for the second peg, etc until there is 1 option for the last peg).

Now imagine that we group together the N holes into M groups, such that the ith group has $f(i)$ holes in that group, and $\sum_i^M f(i) = N$. Also imagine that we want to place the pegs in the holes again, and that the ordering of pegs within each group *does* matter. In that case, the number of ways of arranging these N pegs is the same as if we had no groups in the first place, $N!$.

Let us define C to be the number of ways to arrange the N pegs into M groups such that the ordering of pegs in each group *doesn't* matter. Since there are $f(i)!$ ways of arranging the pegs within group i, then the number of ways of arranging the pegs into these M groups such that the ordering of pegs in each group *does* matter is

$$C \times f(1)! \times f(2)! \times ... \times f(M)! \tag{3.39}$$

But from our previous paragraph, we know this equals $N!$. Solving for C, we get

$$C = \frac{N!}{f(1)! \times f(2)! \times ... \times f(M)!}. \tag{3.40}$$

The entropy of the arrangement of N pegs into M group such that order doesn't matter is

$$S = k_B \ln\Big(\ln N! - \ln f(1)! - \ln f(2)! - ... - \ln f(M)!\Big). \tag{3.41}$$

Now suppose $f(i) \gg 1$ for all i, such that we can use Stirling's formula $\ln N! \approx N \ln N - N$ to simplify the entropy. Using $\sum_i^M f(i) = N$, the N term cancels and we are left with

$$S = N \ln N - \sum_i^M f(i) \ln f(i). \tag{3.42}$$

Since N is a constant, we can ignore it when calculating the entropy, leaving us with

$$S = -\sum_i^M f(i) \ln f(i). \tag{3.43}$$

We now consider a distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$. Each point $\{\boldsymbol{r}, \boldsymbol{v}\}$ in phase space can be thought of as a group of holes, and each particle can be thought of as a peg. We have a known number of particles in each point in phase space, analogous to having a known number of holes in each group. The microstate is the particular arrangement of particles (pegs) which gives us our macrostate $f(\boldsymbol{r}, \boldsymbol{v}, t)$ (the number of holes $f(i)$ in each group).

If we have a known number of pegs $f(i)$ in each group of holes, then the entropy is given by equation 3.43. Therefore, if we have a known number of particles in each point in phase space, $f(\boldsymbol{r}, \boldsymbol{v}) d^3\boldsymbol{r} d^3\boldsymbol{v}$, then the entropy (turning the sum over i into an integral over $\boldsymbol{r}$ and $\boldsymbol{v}$) is eq. (3.37).

3.3 Collision operators

The collision operator $C(f_\sigma)$ is given by

$$C(f_\sigma) = -\Big\langle \frac{q_\sigma}{m_\sigma} \big(\delta \boldsymbol{E} + \boldsymbol{v} \times \delta \boldsymbol{B}\big) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} \delta N_\sigma \Big\rangle. \tag{3.44}$$

In practice, this term cannot be calculated exactly. Instead, various methods for approximating eq. (3.44) are used.

3.3.1 Heuristic estimate of collision operator

We try to get a heuristic estimate of the scaling of $C(f_\sigma)$. We look only at the $\delta \boldsymbol{E} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} \delta N_\sigma$ term and ignore the magnetic field fluctuations. Heuristically,

$$N \sim \frac{\overline{N}}{L^3 v_{\text{th}}^3} \tag{3.45}$$

where $\overline{N}$ is the total number of particles in the system and L is the length scale of the system. From the law of large numbers, $\delta \overline{N} \sim \sqrt{\overline{N}}$ on average for a given microstate. From Gauss's equation

$$\boldsymbol{\nabla} \cdot \delta \boldsymbol{E} = \sum_\sigma \frac{q_\sigma}{\epsilon_0} \int \delta N_\sigma d^3\boldsymbol{v}. \tag{3.46}$$

Because the approximate scale length over which $\boldsymbol{E}$ changes is the Debye length,

$$\boldsymbol{\nabla} \cdot \delta \boldsymbol{E} \sim \delta E / \lambda_D. \tag{3.47}$$

Setting these equal gives

$$\delta E \sim \frac{\lambda_D q V_T^3 \delta N_\sigma}{\epsilon_0} \sim \frac{q\lambda_D \delta \overline{N}}{\epsilon_0 L^3}. \tag{3.48}$$

Plugging these into 3.44 gives a heuristic estimate of $C(f)$:

$$C(f) \sim \delta E \frac{q\delta\overline{N}}{mV_T^4 L^3} \sim \frac{q^2 \lambda_D \delta \overline{N}^2}{\epsilon_0 m V_T^4 L^6} \sim \frac{q^2 \overline{N}}{\epsilon_0 m \omega_P L^3} \frac{1}{L^3 V_T^3} \sim \frac{\omega_P}{L^3 V_T^3}. \tag{3.49}$$

Here $\omega_P^2 = \frac{q^2 n}{\epsilon_0 m}$, $\frac{V_T}{\lambda_D} = \omega_P$ and $\frac{\overline{N}}{L^3} = n$ have been used. We also have that

$$\frac{\partial N}{\partial t} \sim \omega_P N \sim \frac{\omega_P \overline{N}}{L^3 V_T^3}. \tag{3.50}$$

This estimate is for electrostatic fields and relies on Gauss's law. Remember too that in a plasma we only feel the effects of an electrostatic field up to about a Debye length. This means that the system we consider should in fact be a Debye sphere. This means that $\overline{N} \sim \Lambda$. Putting it all together, we have

$$\frac{C(f)}{\partial f/\partial t} \sim \Lambda^{-1} \tag{3.51}$$

If the number of particles in a Debye sphere is much greater than 1, then the collision operator is much smaller than the terms on the left hand side (LHS) of the Vlasov equation. Similarly, if the number of particles in a Debye sphere is less than 1, then the collision operator dominates relative to the LHS terms of the Vlasov equation.

3.3.2 Strongly and weakly coupled plasmas

An ionized gas where the number of particles in a Debye sphere $\Lambda \gg 1$ is a *weakly coupled plasma*. The opposite limit is a *strongly coupled plasma*. The definition of a plasma introduced in section 1 describes a weakly coupled plasma.

Properties of weakly coupled plasmas

Weakly coupled plasmas with $\Lambda \gg 1$ have the following properties. First, the collision operator $C(f)$ is small relative to the terms on the LHS of the Vlasov equation. Second, large-angle collisions are less important than small-angle collisions. Third, the plasma is net neutral over scales larger than a Debye length. Fourth, the kinetic energy per particle, $\frac{1}{2}k_B T_\sigma$, is much greater than the potential energy between any two particles, $\frac{q_\sigma^2}{4\pi\epsilon_0\lambda_D}$. This can be shown with a simple calculation:

$$\frac{KE}{PE} \sim \frac{4\pi\epsilon_0 k_B T_\sigma \lambda_D}{q_\sigma^2} \sim n_\sigma \lambda_D^3 = \Lambda \gg 1. \tag{3.52}$$

Fifth, there is a definite ordering of scale lengths (see section 1.6), such that $\lambda_{mfp} \gg \lambda_D \gg n^{-1/3} \gg b$.

Strongly coupled plasmas

Strongly coupled plasmas with $\Lambda \ll 1$ have the following properties. First, the collision operator in the Vlasov equation dominates relative to the terms on the LHS. Second, large-angle collisions between particles dominate. Third, the plasma is non-neutral. Fourth, the electric potential energy per particle is much larger than the kinetic energy. Fifth, there is a definite ordering of scale length, opposite to that of the weakly-coupled plasmas.

3.3.3 Properties of collision operators

When we have more than one plasma species, it can be convenient to split the collision operator $C(f_\sigma)$ into a linear sum of collision operators between two species,

$$C(f_\sigma) = \sum_\alpha C(f_\sigma, f_\alpha). \tag{3.53}$$

This assumes that only binary (i.e., two-particle) collisions occur. Technically, three-species and four-species collisions are possible, but only considering two-particle collisions is a good approximation when $\Lambda \gg 1$.

Two-particle collision operators should have various properties. Proving that the collision operators we consider in section 3.3.4 do (or do not) satisfy these properties is a homework assignment.

Conservation of particles

Collision operators should satisfy

$$\int C(f_\sigma, f_\alpha)\, d^3\boldsymbol{v} = 0. \tag{3.54}$$

Physically, this means that collisions between particles of species σ and α at some position $\boldsymbol{r}$ only change the velocities of each particle. The number of particles of particles of species σ at position $\boldsymbol{r}$ does not change due to collisions.

Conservation of momentum

Collision operators should satisfy

$$\sum_{\sigma,\alpha} \int m_\sigma \boldsymbol{v} C(f_\sigma, f_\alpha)\, d^3\boldsymbol{v} = 0 \tag{3.55}$$

for all species σ and α. Physically, this means that while particles can exchange momentum between different species, the total momentum at each point $\boldsymbol{r}$ remains constant. In particular, species cannot impart momentum to themselves, which implies that

$$\int m_\sigma \boldsymbol{v} C(f_\sigma, f_\sigma)\, d^3\boldsymbol{v} = 0 \tag{3.56}$$

for all σ.

Conservation of energy

Collision operators should satisfy

$$\sum_{\sigma,\alpha} \int \frac{m_\sigma v^2}{2} C(f_\sigma, f_\alpha)\, d^3\boldsymbol{v} = 0\,. \tag{3.57}$$

Physically, this means that while particles can exchange energy between different species, the total energy at each point $\boldsymbol{r}$ remains constant. If particles were to fuse, releasing atomic energy, this would no longer be strictly true. In particular, species cannot give energy to themselves, which implies that

$$\int \frac{m_\sigma v^2}{2} C(f_\sigma, f_\sigma)\, d^3\boldsymbol{v} = 0 \tag{3.58}$$

for all σ.

Bilinearity

Bilinearity is not a strict assumption, but it is a physically reasonable one. Bilinearity means that for some constants a and b,

$$C(af_\sigma, bf_\alpha) = abC(f_\sigma, f_\alpha) \tag{3.59}$$

for all σ and α. Physically, this means that the frequency of collisions at each point in phase space is proportional to the number of particles at that point in phase space. For example, if we double the number density n in some region, then for electron-electron collisions we have $C(2f_e, 2f_e) = 4C(f_e, f_e)$. This makes sense, because the frequency of collisions is twice as high, and the number of particles is twice as high, so collisions have four times as large an effect.

Locality

The collision operator should in general by local, meaning that $C(f_\sigma, f_\alpha)(\boldsymbol{r}_0, \boldsymbol{v}, t)$ depends on $f_\sigma(\boldsymbol{r}_0, \boldsymbol{v}, t)$ and $f_\alpha(\boldsymbol{r}_0, \boldsymbol{v}, t)$ but not on f at any other value of $\boldsymbol{r}$. Locality also requires that C not depend on any derivatives with respect to positive. C may, however, depend on derivatives with respect to velocity.

Entropy-increasing

$t \to \infty$, collision operators should cause f to approach a Maxwellian velocity distribution, which is the maximum-entropy state.

Positivity-preserving

$C(f)$ should ensure that $f \geq 0$, as f cannot be negative.

3.3.4 Examples of collision operators

We now investigate some of the collisions operators introduced in class.

Krook collision operator

The Krook collision operator

$$C(f) = -\nu(f - f_m). \tag{3.60}$$

This is one of the simplest ways of writing the collision operator. It gives a Maxwellian distribution as t goes to infinity.

Mush limit

The so-called mush limit is

$$C(f) = 0. \tag{3.61}$$

This limit occurs if we take $m_e \to 0$, $e \to 0$, $n \to \infty$, $ne \to$ constant, and $\frac{e}{m} \to$constant. As a result, $\lambda_D \to$ constant, $V_T \to$ constant, and $\omega_p \to$ constant. In this limit, the collision frequency is much lower than the plasma frequency, and the collision operator can be ignored relative to the terms on the LHS of the Vlasov equation.

Diffusion in velocity space

A collision operator corresponding to diffusion in parallel velocity space is

$$C(f) = \frac{\partial}{\partial v_\parallel} D(v_\parallel) \frac{\partial}{\partial v_\parallel} f. \tag{3.62}$$

This is simply a diffusion equation for f with a diffusion coefficient of $D(v_\parallel)$. This operator might represent, for example, the effect of plasma waves causing diffusion of particles in velocity space.

Lorentz collision operator

The form of the Lorentz collision operator introduced in class is

$$\mathcal{L}(f) = \nu(v) \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial}{\partial \mu} f\right]. \tag{3.63}$$

Here $\mu = \frac{v\|}{v} = \cos\theta$, where θ is the pitch angle relative to the magnetic field. μ ranges from -1 to 1. This μ is different from the adiabatic invariant. The collision frequency $\nu(v) \sim \frac{1}{v^3}$.

We do not derive this operator in class or in the homework. It is a simplified version of the Focker-Planck operator, under an assumption that the ions are a cold drifting population. $\nu(v)$ represents the magnitude of the frequency of collisions, while the rest of the operator represents pitch angle scattering of electrons due to collisions with ions in a system where the azimuthal angle with respect to the magnetic field ϕ is negligible. The Lorentz collision operator is, like eq. (3.62) qualitatively similar to the diffusion equation. The Lorentz collision operator is valid under the assumption that $Z_i \gg 1$, so the electron-electron collisions are negligible and only electron-ion collisions are important. It also relies on the assumption that $v_{\text{th}i} \ll v_{\text{th}e}$, as in the derivation the ions are assumed to be a drifting delta function population.

A helpful property of the Lorentz collision operator is that it is self-adjoint. This means that

$$\int d^3\boldsymbol{v} g \mathcal{L}(f) = \int d^3\boldsymbol{v} f \mathcal{L}(g). \tag{3.64}$$

We can prove this property by integrating by parts and using $\int d^3\boldsymbol{v} = \int v^2 dv \int d\phi \int_{-1}^{1} d\mu$. This gives

$$\int d^3\boldsymbol{v} g \mathcal{L}(f) = \int v^2 dv \int d\phi \int_{-1}^{1} \nu(v) g \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} d\mu = -\int v^2 dv \int d\phi \int_{-1}^{1} \nu(v) \frac{\partial g}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} d\mu. \tag{3.65}$$

The boundary term goes to zero because $\mu = \pm 1$ and so $1-\mu^2 = 0$ on the boundary. The final expression in eq. (3.65) is symmetric in f and g, proving that $\mathcal{L}$ is self-adjoint.

The Lorentz operator has other nice properties. It turns out that if P_n is the nth Legendre polynomial, then

$$\mathcal{L}(P_n(\mu)) = -n(n+1)P_n(\mu). \tag{3.66}$$

Since the Legendre polynomials are complete, we can write f as a sum of Legendre polynomials:

$$f(\mu, v, t) = \sum_n P_n(\mu) a_n(v, t). \tag{3.67}$$

We can show that the higher-n, smaller-v components of f pitch-angle scatter faster. Assuming a spatially-homogenous, zero-field plasma, then the Vlasov equation is simply

$$\frac{\partial f}{\partial t} = \mathcal{L}(f). \tag{3.68}$$

Expanding f in terms of the Legendre polynomials, this becomes

$$\sum_n P_n(\mu) \frac{\partial a_n(v,t)}{\partial t} = \sum_n L(P_n(\mu)) a_n(v,t) = -\sum_n n(n+1) P_n(\mu) a_n(v,t). \tag{3.69}$$

Using orthogonality of Legendre polynomials, we can show that

$$\frac{\partial a_n}{\partial t} = -\nu(v) n(n+1) a_n \tag{3.70}$$

which has the solution

$$a_n(v,t) = a_n(v,0) e^{-\nu(v) n(n+1) t}. \tag{3.71}$$

Since $\nu(v) \sim \frac{1}{v^3}$, the larger-$n$, smaller-$v$ particles pitch-angle scatter more quickly under the Lorentz collision operator.

Lorentz Conductivity

We can use the Lorentz collision operator to calculate the conductivity of a plasma. If an external electric field $\boldsymbol{E}$ is applied, the current

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{3.72}$$

is proportional to $\boldsymbol{E}$, and the conductivity σ is the constant of proportionality.[14]

Assume a spatially homogenous plasma with a net electric field $\boldsymbol{E}$. We would expect the magnetic field to influence the current, and that $\sigma_\parallel$ would be different than $\sigma_\perp$. For this calculation, we'll consider an unmagnetized plasma, so that the conductivity σ is a scalar rather than a tensor. Assume that electrons with distribution function f_e collide with a cold, stationary ion population such that $f_i = n_i(\boldsymbol{r})\delta^3(\boldsymbol{v})$. These collisions can be described by the Lorentz collision operator, eq. (3.63). We assume that the ion distribution function does not change and that the electrons carry all the current. Our goal will be to calculate f_e, and use it to calculate the current and thus σ. The Vlasov-Maxwell equation for the electrons is

$$\frac{\partial f_e}{\partial t} - \frac{e}{m}\boldsymbol{E} \cdot \frac{\partial f_e}{\partial \boldsymbol{v}} = \mathcal{L}(f_e) = \nu(v) \frac{\partial}{\partial \mu}\Big[(1-\mu^2)\frac{\partial f_e}{\partial \mu}\Big]. \tag{3.73}$$

Assume a steady-state solution, such that $\frac{\partial}{\partial t} \to 0$. The equilibrium is a balane between electron-ion collisions and the electric field pushing electrons. If the electric field is not too strong, then we expect our equilibrium distribution to be similar to a Maxwellian, with some small departure from the maximum-entropy state. We therefore can write $f = f_m(1+g)$ where g is some arbitrary function and $g \ll 1$ everywhere in phase space. Define the z-direction to point in the direction of $\boldsymbol{E}$. Equation (3.73) then becomes

$$-\frac{e}{m} E_z \frac{\partial [f_m(1+g)]}{\partial v_z} = \mathcal{L}\big(f_m(1+g)\big). \tag{3.74}$$

Since g is small, its derivatives are also small, so

$$\frac{\partial f_m(1+g)}{\partial v_z} \approx \frac{\partial f_m}{\partial v_z} = -\frac{m v_z}{k_B T} f_m. \tag{3.75}$$

Since $C(f_m) = 0$, then

$$C(f_m(1+g)) = C(f_m) + C(f_m g) = f_m C(g). \tag{3.76}$$

Equation (3.74) becomes

$$\frac{e}{k_B T} E_z v_z f_m = f_m \mathcal{L}(g) \tag{3.77}$$

[14]Note that plasmas typically screen electric fields. We assume that an externally applied electric field is applied which is not screened. Note also that currents can arise in a plasma even without an electric field, such as due to the single-particle drifts of the magnetization current.

Canceling f_m and using the definition $\mu = v_\parallel/v$, this becomes

$$\frac{eE_z v\mu}{k_B T} = \nu(v)\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial g}{\partial\mu}\right] \tag{3.78}$$

We'd like to solve for g. We can't solve this exactly, but instead expand g in terms of the Legendre polynomials

$$g = \sum_n a_n(v,t)P_n(\mu). \tag{3.79}$$

Remembering that the larger-n components of f pitch-angle scatter (i.e. equilibrate) faster, we solve only for the $n = 1$ component of g:

$$g \approx a_1 P_1(\mu) = a_1\mu. \tag{3.80}$$

Plugging this into eq. (3.78), we get

$$\frac{eE_z v\mu}{k_B T} = \nu(v)\frac{\partial}{\partial\mu}\left[(1-\mu^2)a_1\right] = -2\nu(v)\mu a_1 \tag{3.81}$$

$$a_1 = -\frac{eE_z v}{2\nu(v)k_B T} \tag{3.82}$$

so

$$g = -\frac{eE_z v\mu}{2\nu k_B T}. \tag{3.83}$$

Using

$$J_z = \int v_z f_e \, d^3\boldsymbol{v} = \int v_z f_m(1+g)\, d^3\boldsymbol{v} = \int v_z f_m g \, d^3\boldsymbol{v}, \tag{3.84}$$

the current J_z is

$$J_z = \frac{e^2E_z}{2k_BT}\int_0^{2\pi} d\phi \int_{-1}^{1} d\mu \int v^2 dv f_m v^2\mu^2 = \frac{2\pi e^2 E_z}{3k_BT}\int_0^\infty \frac{v^4}{\nu(v)} f_m(v)dv \tag{3.85}$$

From $J_z = \sigma E_z$, we have

$$\sigma = \frac{2\pi e^2}{3k_BT}\int_0^\infty \frac{v^4}{\nu(v)} f_m(v)dv \tag{3.86}$$

I won't carry out the integral, but it isn't hard to do in principle, using $\nu(v) \sim \frac{1}{v^3}$. Equation (3.86) gives the Lorentz conductivity of a plasma.

4 Fluid models of plasmas

The small, clean fusion reactor I am considering is NOT describable by MHD. Thank goodness!

SAMUEL COHEN

In the previous section, we introduced and derived a six-dimensional time-dependent partial differential equation called the Vlasov equation. When combined with the Maxwell equations of electromagnetism, the Vlasov-Maxwell system of equations describes the interaction between charged particles and electromagnetic fields in a plasma.

While the Vlasov-Maxwell system is high-fidelity model of a fully ionized plasma (as long as the collision operator is modeled accurately), it is very high-dimensional and thus quite challenging to solve. To perform practical calculations, both analytically and with computational simulations, it is much easier to work with lower-dimensional models.

In this section, we derive three-dimensional time-dependent equations that model the evolution of a plasma. These are called *fluid* models. Fluid models, especially magnetohydrodynamics (MHD), are the most commonly used models of plasmas. General Plasma Physics II (AST552) at Princeton University is almost entirely dedicated to MHD, for which Jeffrey Freidberg's textbook *Ideal MHD* is a standard reference [4].[15]

There are many different fluid models, each of which makes different assumptions. We introduce and derive three such models. First, we derive the multi-fluid equations by taking moments of the Vlasov equation. These equations are underdetermined (more unknowns than equations) unless additional assumptions are made, known as the *closure problem*. Second, we derive the MHD equations by summing the multi-fluid equations over each species, and assuming large length scales and slow frequencies. These equations are also underdetermined; many different closures exist. The third model we introduce is one such closure, called ideal MHD. Ideal MHD is based on three key assumptions related to the collisionality and length scales in the plasma.

4.1 Multi-fluid model

The multi-fluid equations describe each species σ in a plasma as a fluid with density n_σ, mean velocity $\boldsymbol{u}_\sigma \in \mathbb{R}^3$, pressure tensor $\boldsymbol{P}_\sigma \in \mathbb{R}^{3\times 3}$, and heat flux vector $\boldsymbol{Q}_\sigma \in \mathbb{R}^3$. The multi-fluid equations in non-conservative form are

$$\frac{\partial n_\sigma}{\partial t} + \boldsymbol{\nabla}\cdot(n_\sigma \boldsymbol{u}_\sigma) = 0 \tag{4.1a}$$

$$m_\sigma n_\sigma \frac{d\boldsymbol{u}_\sigma}{dt} = q_\sigma n_\sigma(\boldsymbol{E} + \boldsymbol{u}_\sigma \times \boldsymbol{B}) - \boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma + \sum_{\alpha\neq\sigma} \boldsymbol{R}_{\sigma\alpha} \tag{4.1b}$$

$$\frac{3}{2} n_\sigma \frac{dT_\sigma}{dt} = -\boldsymbol{P}_\sigma : \boldsymbol{\nabla}\boldsymbol{u}_\sigma - \boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma + \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha \tag{4.1c}$$

where each time-derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}_\sigma \cdot \boldsymbol{\nabla}$ is called a 'convective' or 'material' derivative. The convective derivative accounts for the fact that the local coordinate frame is stationary but the plasma is moving with velocity $\boldsymbol{u}_\sigma$. In conservative form, eq. (4.1b) and eq. (4.1c) are given by eq. (4.10) and eq. (4.15).

The number density n_σ is the number of particles of species σ per unit volume. The velocity $\boldsymbol{u}_\sigma$ is the mean velocity of particles of species σ. The pressure tensor $\boldsymbol{P}_\sigma$ measures the thermal energy of particles of species σ and is defined in eq. (4.7). The heat flux $\boldsymbol{Q}_\sigma$ measures the direction of thermal energy flow of particles σ and is defined in eq. (4.12). $\boldsymbol{R}_{\sigma\alpha}$ is the force per unit volume imparted to species σ due to collisions with species α and is defined in eq. (4.9). The temperature T_σ is defined as $T_\sigma = \text{Tr}(\boldsymbol{P}_\sigma)/3n_\sigma$ where $\text{Tr}(\boldsymbol{P}_\sigma)$ is the trace of the pressure tensor.[16] $(\frac{\partial W_\sigma}{\partial t})_\alpha$ is the rate of change of the thermal energy per volume of species σ due to collisions with species α and is defined in eq. (4.13).

Physical intuition

The multi-fluid equations each have a straightforward physical interpretation. Equation (4.1a) is simply a continuity equation for species σ, like the continuity equation from electromagnetism $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$. Equation (4.1a) is

[15] To help write this section, I relied on chapters 2 and 9 of [4] as well as chapter 2 of [1]. [2] was also assigned reading, though I did not have time to read it.

[16] In thermodynamics, the definition of temperature is $\frac{1}{T} = \frac{\partial S}{\partial U}$. Strictly speaking, temperature is only well-defined for systems at thermodynamic equilibrium (i.e., at a maximum-entropy state). In plasmas for which $f_\sigma(\boldsymbol{r}, \boldsymbol{v})$ is not Maxwellian at $\boldsymbol{r}$, temperature is technically not defined. Here we define a temperature T_σ that is well-defined even for non-Maxwellian f_σ.

equivalent to a conservation law for particles; it says that if the number of particles n_σ changes inside a volume, it must be due to a flux of particles across the boundary of that region. This can be seen by integrating eq. (4.1a) over an arbitrary volume and using the divergence theorem.

The momentum equation, eq. (4.1b), is simply a statement of Newton's second law. The left hand side (LHS) is the mass (per unit volume) times the acceleration, while the right hand side (RHS) is the force (per unit volume). There are three forces acting on the fluid: the electromagnetic force due to each species being charged, the pressure force due to the thermal motion of particles of species σ, and the friction force representing collisions between particles of species σ and species α.

The energy equation, eq. (4.1c), is simply a statement of conservation of energy. The RHS represents the rate of change of thermal energy,[17] while the LHS represents three processes that change the thermal energy: (1) pressure-volume work, analogous to $-p\,dV$ work in thermodynamics, (2) heat flow, due to thermal diffusion or other convective transport, and (3) energy-transferring collisions with other species.

4.1.1 Assumptions in multi-fluid model

The only approximation made in deriving eqs. (4.1a) to (4.1c) is the assumption that the Vlasov-Maxwell system of equations is a valid description of the plasma. The Vlasov-Maxwell equation (eq. (4.2)) assumes that a large number of discrete charged particles are interacting via electromagnetic forces, and takes the ensemble average thereby replacing a discrete distribution function with a smooth distribution function. Any effects due to the discrete nature of particles (i.e., collisions) are contained in the collision operator $\sum_\alpha C(f_\sigma, f_\alpha)$. Other forces, such as gravity, are ignored.

However, eqs. (4.1a) to (4.1c) have more equations than unknowns, so they cannot be used to solve for the behavior of plasmas unless additional assumptions are made. In section 4.1.3, we consider two possible closures, adiabatic and isothermal. Both of these assume an isotropic distribution function f_σ and either fast or slow changes in the plasma. Another possible closure is the Braginskii closure, described in a classic 1965 paper [2] and still used to in modern plasma codes to study, for example, the scrape-off layer of Tokamaks. In [2], each of the terms in the full energy equation is estimated for a *collisional, magnetized, isotropic* plasma using physical reasoning and heuristic arguments. The result is that $\boldsymbol{Q}_\sigma$ and $(\frac{\partial W_\sigma}{\partial t})_\alpha$ are replaced by so-called 'Braginskii coefficients' which are based on the other plasma variables n_σ, u_σ, and P_σ.

Much of the physics and chemistry of plasmas is ignored by the Vlasov-Maxwell equation and thus the multi-fluid equations. For example, radiation, ionization and excitation, recombination, molecular dissociation and association, fusion, and plasma-boundary interactions are ignored in the multi-fluid model. By including additional terms and/or equations, it is possible to model these effects in numerical simulation codes.

4.1.2 Derivation of multi-fluid equations

To derive the multi-fluid equations, we start with the Vlasov-Maxwell equation

$$\frac{\partial f_\sigma}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\sigma + \frac{q_\sigma}{m_\sigma}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma = \sum_\alpha C(f_\sigma, f_\alpha). \tag{4.2}$$

From this starting point, we'll take *moments* of the distribution function. Moments involve multiplying by a function of a coordinate (typically the coordinate to some power) then integrating over that coordinate. The nth moment of $f(x)$ involves multiplying by x^n, then integrating $\int(\dots)\,dx$.[18] To derive eqs. (4.1a) to (4.1c), we'll take the 0th moment of velocity, 1st moment of velocity, and 2nd moment of velocity of the Vlasov-Maxwell equation.

Derivation: multi-fluid continuity equation

Equation (4.1a) is derived by taking the 0th moment of velocity of eq. (4.2). Thus, we multiply by 1 (since $(\boldsymbol{v})^0 = 1$) and integrate over velocity space. This gives

$$\int \frac{\partial f_\sigma}{\partial t}\,d^3\boldsymbol{v} + \int \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\sigma\,d^3\boldsymbol{v} + \int \frac{q_\sigma}{m_\sigma}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma\,d^3\boldsymbol{v} = \int \sum_\alpha C(f_\sigma, f_\alpha)\,d^3\boldsymbol{v}\,. \tag{4.3}$$

To simplify the first term, take the time-derivative outside of the integral. The first term then becomes, using eq. (3.28), $\frac{\partial}{\partial t}\int f_\sigma\,d^3\boldsymbol{v} = \frac{\partial n_\sigma}{\partial t}$.

[17] The thermal energy in the plasma is $\frac{3}{2}n_\sigma T_\sigma$, whose time-derivative we will show to be (after some simplifications) $\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}$.

[18] In first-semester physics, for example, the so-called 'moment of inertia' was given by $I = \int \rho(r)r^2 dV$ where ρ is mass density and r is the distance from the axis of rotation to the point being integrated. Since I is an integral of ρ times the radius r to the second power, moment of inertia might instead be called the 'second radial moment of mass density'.

To simplify the second term, recall that $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables in the Vlasov description. Thus $\boldsymbol{v} \cdot \boldsymbol{\nabla} f_\sigma = \boldsymbol{\nabla} \cdot (f_\sigma \boldsymbol{v})$, and the second term becomes, using eq. (3.30),

$$\int \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\sigma \, d^3\boldsymbol{v} = \int \boldsymbol{\nabla} \cdot (f_\sigma \boldsymbol{v}) \, d^3\boldsymbol{v} = \boldsymbol{\nabla} \cdot \int f_\sigma \boldsymbol{v} \, d^3\boldsymbol{v} = \boldsymbol{\nabla} \cdot (n_\sigma \boldsymbol{u}_\sigma). \tag{4.4}$$

The third term equals zero. To show this, we first integrate by parts:

$$\int (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma \, d^3\boldsymbol{v} = \int \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (f_\sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})) \, d^3\boldsymbol{v} - \int f_\sigma \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \, d^3\boldsymbol{v} \,. \tag{4.5}$$

The first term becomes a boundary term, which integrates to zero at the boundary at infinity in velocity space because f_σ is zero there. The second term is zero because $\boldsymbol{E}$ doesn't depend on $\boldsymbol{v}$, and the ith component of $(\boldsymbol{v} \times \boldsymbol{B})$ is perpendicular to v_i, so taking the derivative with respect to each component of v_i gives 0.

The term on the RHS is also zero, from particle conservation (eq. (3.54)) for the collision operator C.

With these simplifications, eq. (4.3) becomes eq. (4.1a).

Derivation: multi-fluid momentum equation

Equation (4.1b) is derived by taking the 1st moment of velocity of eq. (4.2). Thus, we multiply by $\boldsymbol{v}$ and integrate over velocity space. This gives

$$\int \frac{\partial f_\sigma}{\partial t} \boldsymbol{v} \, d^3\boldsymbol{v} + \int \boldsymbol{v} (\boldsymbol{v} \cdot \boldsymbol{\nabla} f_\sigma) \, d^3\boldsymbol{v} + \int \frac{q_\sigma}{m_\sigma} \boldsymbol{v} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f_\sigma \, d^3\boldsymbol{v} = \int \sum_\alpha C(f_\sigma, f_\alpha) \boldsymbol{v} \, d^3\boldsymbol{v} \,. \tag{4.6}$$

To simplify the first term, we again pull the time derivative out of the integral and replace $\frac{\partial}{\partial t} \int f_\sigma \boldsymbol{v} \, d^3\boldsymbol{v}$ with $\frac{\partial}{\partial t}(n_\sigma \boldsymbol{u}_\sigma)$.

To simplify the second term, we again begin by pulling the gradient with respect to real space $\boldsymbol{r}$ out of the integral over velocity space $\boldsymbol{v}$. This gives $\int \boldsymbol{\nabla} \cdot (f_\sigma \boldsymbol{v}\boldsymbol{v}) \, d^3\boldsymbol{v}$. We then decompose $\boldsymbol{v}$ into the sum of two terms, the mean fluid velocity $\boldsymbol{u}_\sigma(\boldsymbol{r})$ and the fluctuation of the velocity relative to the mean $\boldsymbol{v}'$. Note that by definition, $\int f_\sigma \boldsymbol{v}' \, d^3\boldsymbol{v} = 0$. Replacing $\boldsymbol{v}$ with $\boldsymbol{u}_\sigma + \boldsymbol{v}'$, the second term becomes

$$\int \boldsymbol{\nabla} \cdot (f_\sigma \boldsymbol{v}\boldsymbol{v}) \, d^3\boldsymbol{v} = \boldsymbol{\nabla} \cdot \int \boldsymbol{u}_\sigma \boldsymbol{u}_\sigma f_\sigma \, d^3\boldsymbol{v}' + \boldsymbol{\nabla} \cdot \int \boldsymbol{u}_\sigma \boldsymbol{v}' f_\sigma \, d^3\boldsymbol{v}' + \boldsymbol{\nabla} \cdot \int \boldsymbol{v}' \boldsymbol{u}_\sigma f_\sigma \, d^3\boldsymbol{v}' + \boldsymbol{\nabla} \cdot \int \boldsymbol{v}' \boldsymbol{v}' f_\sigma \, d^3\boldsymbol{v}' \,.$$

Each of these 4 terms is a divergence of a rank-2 tensor, which gives a vector. We can pull the mean velocity $\boldsymbol{u}_\sigma(\boldsymbol{r})$ out of the velocity integrals. Doing this, the first term term simplifies to $\boldsymbol{\nabla} \cdot (n_\sigma \boldsymbol{u}_\sigma \boldsymbol{u}_\sigma)$, and the second and third terms simplify to zero because they are linear in $\boldsymbol{v}'$. We then define the ijth component of the pressure tensor $\boldsymbol{P}_\sigma$ to be

$$P_{\sigma,ij} = m_\sigma \int f_\sigma v_i' v_j' \, d^3\boldsymbol{v}' \,. \tag{4.7}$$

With this definition, the fourth term becomes $\frac{1}{m_\sigma}$ times the divergence of the pressure tensor. Thus, the second term simplifies to

$$\boldsymbol{\nabla} \cdot (n_\sigma \boldsymbol{u}_\sigma \boldsymbol{u}_\sigma) + \frac{1}{m_\sigma} \boldsymbol{\nabla} \cdot \boldsymbol{P}_\sigma .$$

To simplify the third term, we again integrate by parts, which gives

$$\frac{q_\sigma}{m_\sigma} \int \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot \Big[(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) f_\sigma \boldsymbol{v} \Big] \, d^3\boldsymbol{v} - \frac{q_\sigma}{m_\sigma} \int f_\sigma \boldsymbol{\nabla} \cdot (\boldsymbol{v} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})) \, d^3\boldsymbol{v} \,.$$

This first term becomes a boundary term at infinity in velocity-space which goes to 0 because f_σ goes to zero at infinite velocity. The $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$ can be taken outside the gradient because (as we established before) its gradient with respect to velocity is zero. The third term thus becomes

$$-\frac{q_\sigma}{m_\sigma} \int f_\sigma \big((\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} \big) \boldsymbol{v} \, d^3\boldsymbol{v} \,.$$

In Einstein notation, the ith component of $((\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}})\boldsymbol{v}$ can be written as $(E_j + (\boldsymbol{v} \times \boldsymbol{B})_j) \frac{\partial}{\partial v_j} v_i$. Since $\frac{\partial v_j}{\partial v_i} = \delta_{ij}$, the third term becomes

$$-\frac{q_\sigma}{m_\sigma} \int f_\sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \, d^3\boldsymbol{v} \,.$$

Pulling $\boldsymbol{E}$ and $\boldsymbol{B}$ out of the integral, the third term simplifies to

$$-\frac{q_\sigma n_\sigma}{m_\sigma}(\boldsymbol{E}+\boldsymbol{u}_\sigma\times\boldsymbol{B}).$$

To simplify the term on the RHS, we use eq. (3.56) to set the $\alpha=\sigma$ term to zero. However, the $\alpha\neq\sigma$ are nonzero. The RHS term thus becomes

$$\sum_{\alpha\neq\sigma}\int C(f_\sigma,f_\alpha)\boldsymbol{v}\,d^3\boldsymbol{v}\,.$$

This expression cannot be simplified further unless we know the form of the collision operator. However, as shown in a homework problem in AST568 (Introduction to Classical and Neoclassical Transport and Confinement), for the Lorentz collision operator (eq. (3.63))

$$\int C(f_\sigma,f_\alpha)\boldsymbol{v}\,d^3\boldsymbol{v}=-\nu_{\sigma\alpha}n_\sigma(\boldsymbol{u}_\sigma-\boldsymbol{u}_\alpha)=\frac{1}{m_\sigma}\boldsymbol{R}_{\sigma\alpha}. \tag{4.8}$$

For other collision operators, $\boldsymbol{R}_{\sigma\alpha}$ is defined as

$$\boldsymbol{R}_{\sigma\alpha}=m_\sigma\int C(f_\sigma,f_\alpha)\boldsymbol{v}\,d^3\boldsymbol{v}\,. \tag{4.9}$$

We can interpret $\boldsymbol{R}_{\sigma\alpha}$ as the force per unit volume imparted to species σ due to collisions with species α. Thus, the RHS of equation 4.6 simplifies to

$$\sum_{\alpha\neq\sigma}\frac{1}{m_\sigma}\boldsymbol{R}_{\sigma\alpha}.$$

Having simplified each of the four terms in eq. (4.6), we multiply each by m_σ. Thus eq. (4.6) becomes

$$m_\sigma\frac{\partial(n_\sigma\boldsymbol{u}_\sigma)}{\partial t}+m_\sigma\boldsymbol{\nabla}\cdot(n_\sigma\boldsymbol{u}_\sigma\boldsymbol{u}_\sigma)=q_\sigma n_\sigma\Big(\boldsymbol{E}+\boldsymbol{u}_\sigma\times\boldsymbol{B}\Big)-\boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma+\sum_{\alpha\neq\sigma}\boldsymbol{R}_{\sigma\alpha}. \tag{4.10}$$

This is the momentum equation in *conservative form*. We now show how to write it in non-conservative form. Expanding the LHS of eq. (4.10) gives

$$m_\sigma\boldsymbol{u}_\sigma\frac{\partial n_\sigma}{\partial t}+m_\sigma n_\sigma\frac{\partial\boldsymbol{u}_\sigma}{\partial t}+m_\sigma\boldsymbol{u}_\sigma\boldsymbol{\nabla}\cdot(n_\sigma\boldsymbol{u}_\sigma)+m_\sigma n_\sigma(\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})\boldsymbol{u}_\sigma.$$

The first and third terms are equal to

$$m_\sigma\boldsymbol{u}_\sigma\big(\frac{\partial n_\sigma}{\partial t}+\boldsymbol{\nabla}\cdot(n_\sigma\boldsymbol{u}_\sigma)\big)$$

which is zero due to the continuity equation, eq. (4.1a). This is known as having an *embedded* continuity equation inside eq. (4.10). The second and fourth terms become

$$m_\sigma n_\sigma\big(\frac{\partial\boldsymbol{u}_\sigma}{\partial t}+(\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})\boldsymbol{u}_\sigma\big)=m_\sigma n_\sigma\frac{d\boldsymbol{u}_\sigma}{dt}.$$

Replacing the LHS of eq. (4.10) with $m_\sigma n_\sigma\frac{d\boldsymbol{u}_\sigma}{dt}$ gives the momentum equation, eq. (4.1b).

Derivation: multi-fluid energy equation

Equation (4.1a) is derived by taking the 2nd moment of velocity of eq. (4.2). Rather than multiplying by the tensor $\boldsymbol{v}\boldsymbol{v}$, we multiply by the scalar $\frac{1}{2}m_\sigma(\boldsymbol{v}\cdot\boldsymbol{v})=\frac{1}{2}m_\sigma v^2$. This gives

$$\frac{m_\sigma}{2}\int v^2\frac{\partial f_\sigma}{\partial t}\,d^3\boldsymbol{v}+\frac{m_\sigma}{2}\int v^2\boldsymbol{v}\cdot\boldsymbol{\nabla}f_\sigma\,d^3\boldsymbol{v}+\frac{1}{2}\int q_\sigma v^2(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}f_\sigma\,d^3\boldsymbol{v}=\frac{m_\sigma}{2}\int\sum_\alpha v^2C(f_\sigma,f_\alpha)\,d^3\boldsymbol{v}\,. \tag{4.11}$$

To simplify each term into macroscopic variables, we'll again decompose $\boldsymbol{v}$ into two terms, such that $\boldsymbol{v}=\boldsymbol{u}_\sigma(\boldsymbol{r})+\boldsymbol{v}'$. Once again, we'll eliminate any terms linear in $\boldsymbol{v}'$.

The first term in eq. (4.11) becomes

$$\frac{m_\sigma}{2}\int v^2\frac{\partial f_\sigma}{\partial t}\,d^3\boldsymbol{v}=\frac{m_\sigma}{2}\frac{\partial}{\partial t}\int u_\sigma^2 f_\sigma\,d^3\boldsymbol{v}+\frac{\partial}{\partial t}\int\frac{m_\sigma}{2}v'^2 f_\sigma\,d^3\boldsymbol{v}'=\frac{1}{2}\frac{\partial}{\partial t}\Big(m_\sigma n_\sigma u_\sigma^2+\mathrm{Tr}(\boldsymbol{P}_\sigma)\Big).$$

Defining $\mathrm{Tr}(\boldsymbol{P}_\sigma) = 3n_\sigma T_\sigma$, this becomes

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2 + \frac{3}{2}n_\sigma T_\sigma\Big).$$

To simplify the second term in eq. (4.11), we first pull the gradient out of the integral. Decomposing $\boldsymbol{v}$ and eliminating terms linear in $\boldsymbol{v}'$ then gives

$$\frac{m_\sigma}{2}\boldsymbol{\nabla}\cdot\int\big[u_\sigma^2\boldsymbol{u}_\sigma + v'^2\boldsymbol{v}' + v'^2\boldsymbol{u}_\sigma + 2(\boldsymbol{u}_\sigma\cdot\boldsymbol{v}')\boldsymbol{v}'\big]f_\sigma\, d^3\boldsymbol{v}'\,.$$

The second term in the above expression can be simplified by defining the heat flux

$$\boldsymbol{Q}_\sigma = \int\frac{m_\sigma v'^2}{2}\boldsymbol{v}' f_\sigma\, d^3\boldsymbol{v}' \tag{4.12}$$

while the third and fourth terms can be written in terms of the pressure tensor $\boldsymbol{P}_\sigma$. The second term in eq. (4.11) thus becomes

$$\boldsymbol{\nabla}\cdot\Big[\frac{m_\sigma n_\sigma u_\sigma^2}{2}\boldsymbol{u}_{\boldsymbol{\sigma}} + \boldsymbol{Q}_\sigma + \frac{3}{2}n_\sigma T_\sigma\boldsymbol{u}_\sigma + \boldsymbol{u}_\sigma\cdot\boldsymbol{P}_\sigma\Big].$$

The third term in eq. (4.11) can be rewritten as

$$\frac{q_\sigma}{2}\int v^2\boldsymbol{\nabla}_{\boldsymbol{v}}\cdot\big(f_\sigma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\big)\, d^3\boldsymbol{v}$$

because (as previously discussed) the ith component of $\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}$ doesn't depend on v_i. Integrating by parts, the boundary term goes to zero which leaves

$$-q_\sigma\int f_\sigma\boldsymbol{v}\cdot(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\, d^3\boldsymbol{v} = -q_\sigma n_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{E}.$$

The RHS term in eq. (4.11) can be rewritten as

$$\sum_{\alpha\neq\sigma}\int(u_\sigma^2 + v'^2 + 2\boldsymbol{u}_\sigma\cdot\boldsymbol{v}')C(f_\sigma,f_\alpha)\, d^3\boldsymbol{v}' = 0 + \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha + \sum_{\alpha\neq\sigma}\boldsymbol{u}_\sigma\cdot\boldsymbol{R}_{\sigma\alpha}$$

where

$$\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha \equiv \frac{m_\sigma}{2}\int v'^2 C(f_\sigma,f_\alpha)\, d^3\boldsymbol{v} \tag{4.13}$$

and $\boldsymbol{R}_{\sigma\alpha}$ is defined using eq. (4.9). With these simplifications, eq. (4.11) becomes

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2 + \frac{3}{2}n_\sigma T_\sigma\Big) + \boldsymbol{\nabla}\cdot\Big(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2\boldsymbol{u}_{\boldsymbol{\sigma}} + \frac{3}{2}n_\sigma T_\sigma\boldsymbol{u}_\sigma\Big) = \\ -\boldsymbol{\nabla}\cdot\Big(\boldsymbol{u}_\sigma\cdot\boldsymbol{P}_\sigma\Big) - \boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma + q_\sigma n_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{E} + \sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha + \sum_{\alpha\neq\sigma}\boldsymbol{u}_\sigma\cdot\boldsymbol{R}_{\sigma\alpha}. \tag{4.14}$$

However, there are embedded continuity and momentum equations that will allow us to simplify further. Expanding the first and third terms on the LHS gives

$$\frac{\partial}{\partial t}(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2) + \boldsymbol{\nabla}\cdot(\frac{1}{2}m_\sigma n_\sigma u_\sigma^2\boldsymbol{u}_\sigma) = \frac{m_\sigma u_\sigma^2}{2}(\frac{\partial n_\sigma}{\partial t} + \boldsymbol{\nabla}\cdot(n_\sigma\boldsymbol{u}_\sigma)) + \frac{m_\sigma n_\sigma}{2}(\frac{\partial u_\sigma^2}{\partial t} + (\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})u_\sigma^2).$$

Setting the embedded continuity equation to zero, the first and third LHS terms in eq. (4.14) become

$$\frac{m_\sigma n_\sigma}{2}(\frac{\partial u_\sigma^2}{\partial t} + (\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})u_\sigma^2).$$

Expanding again, we have

$$\frac{\partial u_\sigma^2}{\partial t} = 2\boldsymbol{u}_\sigma\cdot\frac{\partial\boldsymbol{u}_\sigma}{\partial t}$$

and (using Einstein notation)

$$(\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})u_\sigma^2 = \sum_j(u_{\sigma i}\frac{\partial}{\partial r_i})u_{\sigma j}^2 = 2u_{\sigma j}u_{\sigma i}\frac{\partial u_{\sigma j}}{\partial r_i} = 2\boldsymbol{u}_\sigma\cdot[(\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})\boldsymbol{u}_\sigma].$$

These terms then become

$$\frac{m_\sigma n_\sigma}{2}\Big(2\boldsymbol{u}_\sigma\cdot\frac{\partial\boldsymbol{u}_\sigma}{\partial t}+2\boldsymbol{u}_\sigma\cdot[(\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla})\boldsymbol{u}_\sigma]\Big)=\boldsymbol{u}_\sigma\cdot\Big(m_\sigma n_\sigma\frac{d\boldsymbol{u}_\sigma}{dt}\Big).$$

This gives the LHS of the momentum equation dotted with $\boldsymbol{u}_\sigma$. Replacing this with the RHS of the momentum equation gives

$$\boldsymbol{u}_\sigma\cdot\Big(q_\sigma n_\sigma\boldsymbol{E}+q_\sigma n_\sigma\boldsymbol{u}_\sigma\times\boldsymbol{B}-\boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma+\sum_{\alpha\neq\sigma}\boldsymbol{R}_{\sigma\alpha}\Big)=q_\sigma n_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{E}+\sum_{\alpha\neq\sigma}\boldsymbol{u}_\sigma\cdot\boldsymbol{R}_{\sigma\alpha}-\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma.$$

The $q_\sigma n_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{E}$ and $\sum_{\alpha\neq\sigma}\boldsymbol{u}_\sigma\cdot\boldsymbol{R}_{\sigma\alpha}$ terms cancel with the third and fifth terms on the RHS of eq. (4.14). With these cancellations, eq. (4.11) becomes

$$\frac{\partial}{\partial t}\Big(\frac{3}{2}n_\sigma T_\sigma\Big)+\boldsymbol{\nabla}\cdot\Big(\frac{3}{2}n_\sigma T_\sigma\boldsymbol{u}_\sigma\Big)=\boldsymbol{u}_\sigma\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma)-\boldsymbol{\nabla}\cdot\Big(\boldsymbol{u}_\sigma\cdot\boldsymbol{P}_\sigma\Big)-\boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma+\sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha.$$

Using Einstein summation notation,

$$\boldsymbol{u}_\sigma\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{P}_\sigma)-\boldsymbol{\nabla}\cdot(\boldsymbol{u}_\sigma\cdot\boldsymbol{P}_\sigma)=u_{\sigma i}\frac{\partial}{\partial r_j}P_{\sigma ij}-\frac{\partial}{\partial r_i}(u_{\sigma j}P_{\sigma ij})=$$
$$u_{\sigma i}\frac{\partial}{\partial r_j}P_{\sigma ij}-u_{\sigma j}\frac{\partial}{\partial r_i}P_{\sigma ij}-P_{\sigma ij}\frac{u_{\sigma j}}{\partial r_i}=-P_{\sigma ij}\frac{u_{\sigma j}}{\partial r_i}=-\boldsymbol{P}_\sigma:\boldsymbol{\nabla}\boldsymbol{u}_\sigma.$$

The $u_{\sigma i}\frac{\partial}{\partial r_j}P_{\sigma ij}$ and $-u_{\sigma j}\frac{\partial}{\partial r_i}P_{\sigma ij}$ terms cancel due to the symmetry of $\boldsymbol{P}_\sigma$. The : symbol is the tensor inner product, and it means summation over both indices. Equation (4.11) then becomes

$$\frac{\partial}{\partial t}\Big(\frac{3}{2}n_\sigma T_\sigma\Big)+\boldsymbol{\nabla}\cdot\Big(\frac{3}{2}n_\sigma T_\sigma\boldsymbol{u}_\sigma\Big)=-\boldsymbol{P}_\sigma:\boldsymbol{\nabla}\boldsymbol{u}_\sigma-\boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma+\sum_{\alpha\neq\sigma}\Big(\frac{\partial W_\sigma}{\partial t}\Big)_\alpha. \tag{4.15}$$

This is the energy equation in *conservative form*. To write it in non-conservative form, we expand the LHS and use an embedded continuity equation:

$$\frac{\partial}{\partial t}\Big(\frac{3}{2}n_\sigma T_\sigma\Big)+\boldsymbol{\nabla}\cdot\Big(\frac{3}{2}n_\sigma T_\sigma\boldsymbol{u}_\sigma\Big)=\frac{3}{2}\frac{\partial n_\sigma}{\partial t}T_\sigma+\frac{3}{2}\boldsymbol{\nabla}\cdot(n_\sigma\boldsymbol{u}_\sigma)T_\sigma+\frac{3}{2}n_\sigma\frac{\partial T_\sigma}{\partial t}+\frac{3}{2}n_\sigma\boldsymbol{u}_\sigma\cdot\boldsymbol{\nabla}T_\sigma=\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}.$$

Replacing the LHS of eq. (4.15) with $\frac{3}{2}n_\sigma\frac{dT_\sigma}{dt}$ gives eq. (4.1c).

4.1.3 Closure of multi-fluid equations

To derive the continuity equation, we took the zeroth moment of eq. (4.2). This gave us an equation for the time-evolution of n_σ, the zeroth moment of f_σ (see eq. (3.28)), which depends on $\boldsymbol{u}_\sigma$, the first moment of f_σ (see eq. (3.30)). Thus, solving for $n_\sigma(\boldsymbol{r},t)$ requires knowing $\boldsymbol{u}_\sigma(\boldsymbol{r},t)$. Fortunately, by taking the first moment of eq. (4.2), we get an equation for the time-derivative of $\boldsymbol{u}_\sigma$. Unfortunately, this equation depends on $\boldsymbol{P}_\sigma$, the second moment of f_σ. We can get an equation for time-derivative of (the trace of) $\boldsymbol{P}_\sigma$ by taking the second moment of eq. (4.2), but unfortunately this depends on a third moment of f_σ, $\boldsymbol{Q}_\sigma$.

The fact that fluid equations derived from kinetic equations always have more unknown variables than equations is called the closure problem. It stems from the fact that the distribution function f_σ has infinite possible degrees of freedom in the velocity distribution, while fluid equations only capture a finite number of those degrees of freedom (zeroth moment, first moment, second moment, etc). The result is that fluid equations cannot be solved exactly, unless some assumption(s) about the higher-order moments of f_σ are made.

In the rest of this subsection, we will look at two ways of closing the multi-fluid equations – isothermal closure and adiabatic closure – both of which assume isotropy of the distribution function.

Isotropy assumption

The pressure tensor $\boldsymbol{P}_\sigma$ can be written as the sum of an isotropic term $P_\sigma\boldsymbol{I}$ and an anisotropic term, $\boldsymbol{\Pi}_\sigma$,

$$\boldsymbol{P}_\sigma=P_\sigma\boldsymbol{I}+\boldsymbol{\Pi}_\sigma \tag{4.16}$$

where $\boldsymbol{I}$ is the identity matrix.

If we assume that our plasma has an isotropic velocity distribution, then we can set the anisotropic component $\boldsymbol{Pi}_\sigma = 0$ and replace the pressure tensor $\boldsymbol{P}_\sigma$ with a scalar P_σ. Isotropy means that the distribution function f_σ is identical in every direction, so that $f_\sigma(\boldsymbol{v}) = f_\sigma(v)$ where v is the magnitude of $\boldsymbol{v}$. For isotropic systems,

$$P_\sigma = \int m_\sigma f_\sigma v_i'^2 \, d^3\boldsymbol{v} = \frac{1}{3}\int m_\sigma f_\sigma v'^2 \, d^3\boldsymbol{v} = n_\sigma T_\sigma. \tag{4.17}$$

With this assumption, the momentum equation becomes

$$m_\sigma n_\sigma \frac{d\boldsymbol{u}_\sigma}{dt} = q_\sigma n_\sigma \boldsymbol{E} + q_\sigma n_\sigma \boldsymbol{u}_\sigma \times \boldsymbol{B} - \boldsymbol{\nabla} P_\sigma + \sum_{\alpha \neq \sigma} \boldsymbol{R}_{\sigma\alpha}. \tag{4.18}$$

The energy equation in conservative form (eq. (4.15) simplifies as well. Assuming isotropy, $\boldsymbol{P}_\sigma : \boldsymbol{\nabla}\boldsymbol{u}_\sigma = P_\sigma \boldsymbol{\nabla}\cdot\boldsymbol{u}_\sigma$. Using $n_\sigma T_\sigma = P_\sigma$, eq. (4.15) then becomes

$$\frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}(\boldsymbol{u}_\sigma \cdot \boldsymbol{\nabla})P_\sigma + \frac{5}{2}P_\sigma \boldsymbol{\nabla}\cdot\boldsymbol{u}_\sigma = -\boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma + \sum_{\alpha\neq\sigma}\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha. \tag{4.19}$$

The assumption of isotropy alone is not enough to close the multi-fluid equations, because the $\boldsymbol{\nabla}\cdot\boldsymbol{Q}_\sigma$ term remains. We'll consider two limits in which the multi-fluid equations can be closed: the adiabatic limit, corresponding to zero heat flow $\boldsymbol{Q} = 0$ and fast changes, and the isothermal limit, corresponding to constant temperature and slow changes.

Adiabatic closure

In the adiabatic limit, changes in the plasma happen fast enough that no heat flows and $\boldsymbol{Q}_\sigma = 0$. Physically, this means that heat flux and collisional energy transfer terms in the energy equation (eq. (4.1c)) are negligible, and only pressure-volume work contributes to changing the temperature.

The adiabatic limit occurs when the velocity at which changes in the plasma occur are fast relative to the thermal velocity $v_{\text{th}\sigma}$. We can see this by looking at the scaling of the LHS and RHS terms in eq. (4.19). The LHS terms scale as $U_{ph}P/L$ where U is the phase velocity of disturbances in the plasma, P is the pressure, and L is a length scale over which the quantities in the plasma vary. Assuming collisions are negligible, the RHS terms are dominated by the heat flux which goes like $\boldsymbol{\nabla}\cdot\int m_\sigma v'^2 \boldsymbol{v}' f_\sigma \, d^3\boldsymbol{v}' \sim P v_{\text{th}\sigma}/L$. Thus, the heat flux is negligible relative to the other terms if $U_{ph} \gg v_{\text{th}\sigma}$.

In the adiabatic limit, the energy equation (eq. (4.19)) becomes

$$3\frac{dP_\sigma}{dt} = -5P_\sigma \boldsymbol{\nabla}\cdot\boldsymbol{u}_\sigma.$$

The continuity equation gives

$$\frac{\partial n_\sigma}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}n_\sigma + n_\sigma\boldsymbol{\nabla}\cdot\boldsymbol{u}_\sigma = 0,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_\sigma = -\frac{1}{n_\sigma}\frac{dn_\sigma}{dt}. \tag{4.20}$$

Combining these gives

$$\frac{1}{P_\sigma}\frac{dP_\sigma}{dt} = \frac{5}{3}\frac{1}{n_\sigma}\frac{dn_\sigma}{dt} \tag{4.21}$$

Setting $\gamma = \frac{5}{3}$, the heat capacity ratio, then eq. (4.21) has the solution

$$\frac{d}{dt}\left(\frac{P_\sigma}{n_\sigma^\gamma}\right) = 0. \tag{4.22}$$

Thus, in the adiabatic limit the energy equation simplifies to an equation for P_σ in terms of n_σ. This allows us to close the multi-fluid equations.

Isothermal closure

In the isothermal limit, changes in the plasma happen slowly enough that heat is able to flow such that the temperature is spatially constant and $\boldsymbol{\nabla}T_\sigma = 0$.

In the isotropic isothermal limit,

$$\boldsymbol{\nabla}P_\sigma = \boldsymbol{\nabla}(n_\sigma T_\sigma) = T_\sigma\boldsymbol{\nabla}n_\sigma.$$

Thus, the $\boldsymbol{\nabla}P_\sigma$ term in eq. (4.18) can be written in terms of n_σ, giving a closed set of equations for n_σ and $\boldsymbol{u}_\sigma$. T_σ is given as an initial or boundary condition. The isothermal condition is often written as

$$\boldsymbol{\nabla}\left(\frac{P_\sigma}{n_\sigma}\right) = 0. \tag{4.23}$$

4.2 Generalized magnetohydrodynamics (MHD)

Magnetohydrodynamics (MHD) is a model to describe the behavior of plasmas that occur at low frequencies and large spatial scales. In this section, we introduce and derive the generalized MHD equations. These equations are not closed (i.e., more unknowns than equations) but are the basis for various MHD closures. In section 4.3, we'll introduce and derive one such closure called *ideal MHD*.

While the multi-fluid equations use density $n_\sigma(\boldsymbol{r})$, velocity $\boldsymbol{u}_\sigma(\boldsymbol{r})$, and pressure $\boldsymbol{P}_\sigma(\boldsymbol{r})$ to describe the state of each plasma species, the MHD equations use a single mass density ρ, a single mean fluid velocity $\boldsymbol{u}$, a current density $\boldsymbol{J}$, and electron and ion pressures $\boldsymbol{P}_e$ and $\boldsymbol{P}_i$. The mass density ρ is defined as

$$\rho = \sum_\sigma m_\sigma n_\sigma. \tag{4.24}$$

The mean fluid velocity $\boldsymbol{u}$ is defined as

$$\boldsymbol{u} = \frac{1}{\rho}\sum_\sigma n_\sigma m_\sigma \boldsymbol{u}_\sigma. \tag{4.25}$$

Likewise, the current density $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} = \sum_\sigma q_\sigma n_\sigma \boldsymbol{u}_\sigma. \tag{4.26}$$

For simplicity, we will assume that our plasma is made up of only two species: electrons, of charge $q_e = -e$ and mass m_e, and ions, of charge $q_i = +Z_i e$ and mass m_i with $m_i \gg m_e$. It is straightforward to derive MHD equations for the case in which more than two species of charged particles are in the plasma. For an ion-electron plasma, the MHD equations are given by the continuity equation for ρ

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{4.27a}$$

the momentum equation

$$\rho\frac{d\boldsymbol{u}}{dt} = \boldsymbol{J}\times\boldsymbol{B} - \boldsymbol{\nabla}\cdot(\boldsymbol{P}_i + \boldsymbol{P}_e), \tag{4.27b}$$

the generalized Ohm's law for MHD

$$\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} + \frac{1}{en_e}\boldsymbol{\nabla}\cdot\boldsymbol{P}_e - \frac{1}{en_e}\boldsymbol{J}\times\boldsymbol{B} = \eta\boldsymbol{J} \tag{4.27c}$$

where the viscosity $\eta = \frac{\nu_{ei} m_e}{e^2 n_e}$, and the MHD energy equations

$$\frac{d}{dt}\left(\frac{P_i}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[\left(\frac{\partial W_i}{\partial t}\right)_e - \boldsymbol{\nabla}\cdot\boldsymbol{Q}_i - \boldsymbol{\Pi}_i : \boldsymbol{\nabla}\boldsymbol{u}\right] \tag{4.27d}$$

$$\frac{d}{dt}\left(\frac{P_e}{\rho^\gamma}\right) = \frac{2}{3\rho^\gamma}\left[\left(\frac{\partial W_e}{\partial t}\right)_i - \boldsymbol{\nabla}\cdot\boldsymbol{Q_e} - \boldsymbol{\Pi}_e : \boldsymbol{\nabla}(\boldsymbol{u} - \frac{\boldsymbol{J}}{en_e})) + \frac{1}{en_e}\boldsymbol{J}\cdot\boldsymbol{\nabla}\left(\frac{P_e}{\rho^\gamma}\right)\right]. \tag{4.27e}$$

Π_i and Π_e are the anisotropic components of $\boldsymbol{P}_i$ and $\boldsymbol{P}_e$, using the decomposition $\boldsymbol{P}_\sigma = P_\sigma \boldsymbol{I} + \boldsymbol{\Pi}_\sigma$. The convective derivative operator $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. Maxwell's equations are also used in MHD, though $\boldsymbol{E}$ is determined by eq. (4.27c) rather than Gauss's law and the displacement current term in Ampere's law is ignored due to the low-frequency assumption of MHD:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{4.28a}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla}\times\boldsymbol{E} \tag{4.28b}$$

$$\mu_0\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B}. \tag{4.28c}$$

Equation (4.28c) also implies a divergence-free condition on $\boldsymbol{J}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0. \tag{4.28d}$$

4.2.1 Assumptions in generalized MHD

There are two assumptions in the generalized MHD model called *asymptotic assumptions*. The first asymptotic assumption is the limit that

$$\epsilon_0 \to 0. \tag{4.29}$$

Because λ_D is proportional to $\sqrt{\epsilon_0}$ (see eq. (1.23)) this implies that $\lambda_D \to 0$. Physically, then, the first asymptotic assumption can be interpreted to mean that the characteristic length scales in the plasma are much larger than the Debye length:

$$\frac{\lambda_D}{L} \ll 1. \tag{4.30}$$

The second asymptotic assumption is the limit that

$$m_e \to 0. \tag{4.31}$$

Because $\Omega_e = |e||B|/m_e$ and ω_{pe} are both inversely proportional to m_e, this implies that the electron cyclotron frequency $\Omega_e \to \infty$ and the electron plasma frequency $\omega_{pe} \to \infty$. Physically, then, the second asymptotic assumption can be interpreted to mean that the characteristic frequencies in the plasma are much smaller than both Ω_e and ω_{pe}:

$$\frac{\omega}{\omega_{pe}} \ll 1 \text{ and } \frac{\omega}{\Omega_e} \ll 1. \tag{4.32}$$

4.2.2 Derivation of generalized MHD equations

The MHD equations is derived by summing or subtracting the multi-fluid equations over each species σ, and eliminating terms which are negligible under the asymptotic assumptions of low frequencies and large spatial scales.

Derivation of Maxwell's equations for MHD

Gauss's law is given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}$$

where $\rho_e = \sum_\sigma q_\sigma n_\sigma$ is the charge density. Due to the first asymptotic assumption that $\epsilon_0 \to 0$, then $\rho_e \to 0$ as well to ensure that $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ is not infinite. $\rho_e \to 0$ implies that in the MHD approximation, plasmas are net neutral. For a two-species plasma of ions and electrons, this implies that

$$\rho_e = \sum_\sigma q_\sigma n_\sigma = Z_i e n_i - e n_e = 0. \tag{4.33}$$

In MHD, $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ is not specified by Gauss's law but instead is determined by the MHD Omh's law, eq. (4.27c). Gauss's law, however, implies that plasmas are electrically neutral in the MHD approximation.

Ampere's law is given by

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Due to the first asymptotic assumption that $\epsilon_0 \to 0$, then the displacement current is negligible in the MHD approximation. As a result, Ampere's law becomes eq. (4.28c).

Derivation of MHD continuity equation

To derive the MHD continuity equation, we multiply the multi-fluid continuity equation eq. (4.1a) by m_σ and sum over species σ:

$$\frac{\partial}{\partial t}(\sum_\sigma m_\sigma n_\sigma) + \boldsymbol{\nabla} \cdot (\sum_\sigma m_\sigma n_\sigma \boldsymbol{u}_\sigma) = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0.$$

This is the MHD continuity equation, given by eq. (4.27a). We have used the definitions for ρ and $\boldsymbol{u}$, given by eqs. (4.24) and (4.25).

Derivation of divergence-free current condition

To derive the condition that the current $\boldsymbol{J}$ is divergence-free, multiply the multi-fluid continuity equation eq. (4.1a) by q_σ and sum over σ:

$$\frac{\partial}{\partial t}\big(\sum_\sigma q_\sigma n_\sigma\big) + \boldsymbol{\nabla} \cdot \big(\sum_\sigma q_\sigma n_\sigma \boldsymbol{u}_\sigma\big) = \frac{\partial \rho_e}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0.$$

This is the continuity equation for charged particles. However, due to the condition that $\rho_e \to 0$, then this simplifies to eq. (4.28d). We can also derive $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ by taking the divergence of eq. (4.28c).

Derivation of MHD momentum equation

To derive the MHD momentum equation, we sum the multi-fluid momentum equation (in conservative form, eq. (4.10)) over σ:

$$\sum_\sigma \frac{\partial(m_\sigma n_\sigma \boldsymbol{u}_\sigma)}{\partial t} + \sum_\sigma \boldsymbol{\nabla}\cdot(m_\sigma n_\sigma \boldsymbol{u}_\sigma \boldsymbol{u}_\sigma) = \sum_\sigma q_\sigma n_\sigma \Big(\boldsymbol{E} + \boldsymbol{u}_\sigma \times \boldsymbol{B}\Big) - \sum_\sigma \boldsymbol{\nabla}\cdot \boldsymbol{P}_\sigma + \sum_\sigma \sum_{\alpha\neq\sigma} \boldsymbol{R}_{\sigma\alpha}. \tag{4.34}$$

The first term on the LHS simplifies (using eq. (4.25)) to

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}).$$

The second term on the LHS doesn't have an exact simplification, due to the $\boldsymbol{u}_\sigma \boldsymbol{u}_\sigma$ term. However, using the second asymptotic assumption that $m_e \to 0$, and (for simplicity) only assuming a single ion species, then

$$\rho = \sum_\sigma m_\sigma n_\sigma \approx m_i n_i$$

and

$$\boldsymbol{u} = \frac{1}{\rho}\sum_\sigma m_\sigma n_\sigma \boldsymbol{u}_\sigma \approx \boldsymbol{u}_i.$$

Thus, the second term on the LHS becomes

$$\boldsymbol{\nabla}\cdot(m_i n_i \boldsymbol{u}_i \boldsymbol{u}_i) \approx \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}\boldsymbol{u}).$$

The first term on the RHS is zero because

$$\sum_\sigma q_\sigma n_\sigma \boldsymbol{E} = \rho_e \boldsymbol{E} \approx 0$$

due to the condition that $\rho_e \to 0$.

The second term on the RHS becomes

$$\sum_\sigma q_\sigma n_\sigma \boldsymbol{u}_\sigma \times \boldsymbol{B} = \boldsymbol{J}\times\boldsymbol{B}$$

using eq. (4.26).

Assuming (for simplicity) only two species, ions and electrions, the third term on the RHS becomes

$$-\boldsymbol{\nabla}\cdot(\boldsymbol{P}_i + \boldsymbol{P}_e).$$

The fourth term on the RHS is zero, because the collisional force $\boldsymbol{R}_{\sigma\alpha}$ is antisymmetric under exchange of σ and α due to Newton's third law:

$$\sum_\sigma \sum_{\alpha\neq\sigma} \boldsymbol{R}_\sigma \alpha = \boldsymbol{R}_{ei} + \boldsymbol{R}_{ie} = 0.$$

With these simplifications, eq. (4.34) becomes

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}\boldsymbol{u}) = \boldsymbol{J}\times\boldsymbol{B} - \boldsymbol{\nabla}\cdot(\boldsymbol{P}_i + \boldsymbol{P}_e). \tag{4.35}$$

This is the MHD momentum equation in conservative form. By expanding the LHS and removing the embedded continuity equation, eq. (4.35) becomes eq. (4.27b).

Derivation of MHD Ohm's law

To derive MHD ohm's law, we divide the multi-fluid momentum equation for electrons eq. (4.1b) by n_e:

$$m_e \frac{d\boldsymbol{u}_e}{dt} = -e\boldsymbol{E} - e\boldsymbol{u}_e \times \boldsymbol{B} - \frac{1}{n_e}\boldsymbol{\nabla}\cdot \boldsymbol{P}_e + \frac{1}{n_e}\sum_{\alpha\neq e} \boldsymbol{R}_{e\alpha}. \tag{4.36}$$

We again assume (for simplicity) a plasma made of two species, ions and electrons. Using eq. (4.33), we have $n_i = n_e/Z_i$. Using

$$\boldsymbol{J} = Z_i e n_i \boldsymbol{u}_i - e n_e \boldsymbol{u}_e$$

we have

$$\boldsymbol{u}_e = -\frac{\boldsymbol{J}}{e n_e} + \boldsymbol{u}_i.$$

We assume that $\boldsymbol{R}_{e\alpha}$ is given by the Lorentz collision operator, so (using eq. (4.8))

$$\sum_{\alpha} \boldsymbol{R}_{e\alpha} = -\sum_{\alpha \neq e} m_e n_e \nu_{e\alpha}(\boldsymbol{u}_e - \boldsymbol{u}_\alpha) = -m_e n_e \nu_{ei}(\boldsymbol{u}_e - \boldsymbol{u}_i) = \frac{m_e \nu_{ei}}{e} \boldsymbol{J}.$$

Plugging these into electron momentum equation, eq. (4.36) becomes

$$m_e \frac{d\boldsymbol{u}_e}{dt} = -e\boldsymbol{E} + \frac{1}{n_e} \boldsymbol{J} \times \boldsymbol{B} - e\boldsymbol{u}_i \times \boldsymbol{B} - \frac{1}{n_e} \boldsymbol{\nabla} \cdot \boldsymbol{P}_e + \frac{m_e \nu_{ei}}{e n_e} \boldsymbol{J}. \tag{4.37}$$

Due to the second asymptotic assumption $\boldsymbol{u}_i \approx \boldsymbol{u}$. Due to the first asymptotic assumption that $m_e \to 0$, we can eliminate the LHS term. Another way to justify this elimination is to use the scaling $m_e \frac{d\boldsymbol{u}_e}{dt} \sim \omega m_e u_e$ and $e\boldsymbol{u}_e \times \boldsymbol{B} \sim e u_e B$ so

$$\frac{m_e \frac{d\boldsymbol{u}_e}{dt}}{e\boldsymbol{u}_e \times \boldsymbol{B}} \sim \frac{\omega m_e}{eB} = \frac{\omega}{\Omega_e} \ll 1$$

which justifies ignoring the electron inertia term relative to the other terms in the electron momentum equation. We also divide by e and set

$$\eta = \frac{m_e \nu_{ei}}{e^2 n}.$$

With these simplifications, eq. (4.37) becomes the generalized MHD Ohm's law eq. (4.27c).

Derivation of MHD energy equation

To derive the MHD energy equations, we start with the multi-fluid energy equation in conservation form, eq. (4.15). Expanding the LHS, we have

$$\frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}\boldsymbol{\nabla} \cdot (P_\sigma \boldsymbol{u}_\sigma) = \frac{3}{2}\frac{\partial P_\sigma}{\partial t} + \frac{3}{2}\boldsymbol{u}_\sigma \cdot \boldsymbol{\nabla} P_\sigma + \frac{3}{2} P_\sigma \boldsymbol{\nabla} \cdot \boldsymbol{u}_\sigma = \frac{3}{2}\frac{dP_\sigma}{dt} + \frac{3}{2} P_\sigma \boldsymbol{\nabla} \cdot \boldsymbol{u}_\sigma.$$

Expanding $\boldsymbol{P}_\sigma = P_\sigma \boldsymbol{I} + \boldsymbol{\Pi}_\sigma$, the first term on the RHS becomes

$$-\boldsymbol{P}_\sigma : \boldsymbol{\nabla} \boldsymbol{u}_\sigma = -P_\sigma \boldsymbol{\nabla} \cdot \boldsymbol{u}_\sigma - \boldsymbol{\Pi}_\sigma : \boldsymbol{u}_\sigma.$$

The multi-fluid energy equation then becomes

$$\frac{3}{2}\frac{dP_\sigma}{dt} + \frac{5}{2} P_\sigma \boldsymbol{\nabla} \cdot \boldsymbol{u}_\sigma = -\boldsymbol{\Pi}_\sigma : \boldsymbol{\nabla} \boldsymbol{u}_\sigma - \boldsymbol{\nabla} \cdot \boldsymbol{Q}_\sigma + \sum_{\alpha \neq \sigma} \left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha.$$

Using eq. (4.20), we set

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_\sigma = -\frac{1}{n_\sigma}\frac{dn_\sigma}{dt}.$$

Setting $\gamma = 5/3$, dividing by $\frac{3}{2}\rho^\gamma$, we have

$$\frac{1}{\rho^\gamma}\frac{dP_\sigma}{dt} - \frac{\gamma}{\rho^\gamma}\frac{P_\sigma}{n_\sigma}\frac{dn_\sigma}{dt} = \frac{2}{3\rho^\gamma}\Big[\sum_{\alpha \neq \sigma}\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha - \boldsymbol{\nabla} \cdot \boldsymbol{Q}_\sigma - \boldsymbol{\Pi}_\sigma : \boldsymbol{\nabla} \boldsymbol{u}_\sigma\Big]. \tag{4.38}$$

Assuming once again, for simplicity, a plasma made of two species (ions and electrons), using $n_e = Z_i n_i$ and $\rho \approx m_i n_i$, we can write the RHS of eq. (4.38) for both $\sigma = i$ and $\sigma = e$ as

$$\frac{1}{\rho^\gamma}\frac{dP_\sigma}{dt} - \frac{\gamma}{\rho^\gamma}\frac{P_\sigma}{m_i n_i}\frac{d(m_i n_i)}{dt} = \frac{d}{dt}\Big(\frac{P_\sigma}{\rho^\gamma}\Big).$$

Note that the $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}_\sigma \cdot \boldsymbol{\nabla}$ operator contains the multi-fluid variable $\boldsymbol{u}_\sigma$, rather than the MHD variables $\boldsymbol{u}$ and $\boldsymbol{J}$. The energy equation for ions is now

$$\frac{d}{dt}\Big(\frac{P_i}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_i}{\partial t}\Big)_e - \boldsymbol{\nabla} \cdot \boldsymbol{Q}_i - \boldsymbol{\Pi}_i : \boldsymbol{\nabla} \boldsymbol{u}_i\Big].$$

Using $\boldsymbol{u}_i \approx \boldsymbol{u}$ due to the second asymptotic assumption, this becomes the MHD energy equation for ions, eq. (4.27d).

For electrons, the energy equation is now

$$\frac{d}{dt}\Big(\frac{P_e}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[\Big(\frac{\partial W_e}{\partial t}\Big)_i - \boldsymbol{\nabla} \cdot \boldsymbol{Q}_e - \boldsymbol{\Pi}_e : \boldsymbol{\nabla} \boldsymbol{u}_e\Big]. \tag{4.39}$$

Replacing $\boldsymbol{u}_e$ with $\boldsymbol{u}$ and $\boldsymbol{J}$ using

$$\boldsymbol{u}_e = \boldsymbol{u}_i - \frac{\boldsymbol{J}}{e n_e} \approx \boldsymbol{u} - \frac{\boldsymbol{J}}{e n_e},$$

eq. (4.39) becomes the MHD energy equation for eletrons, eq. (4.27e).

4.3 Ideal MHD

The most commonly used form of the MHD equations is called ideal MHD. The ideal MHD model is one way of closing the generalized MHD equations. Entire textbooks, in particular [4], are dedicated to the ideal MHD model. General Plasma Physics II (AST552) at Princeton University is mostly focused on MHD, especially ideal MHD.

Ideal MHD uses the same plasma variables ρ, $\boldsymbol{u}$, and $\boldsymbol{J}$ as with generalized MHD, except only a single scalar pressure P is used. In ideal MHD, P is defined by

$$P = \sum_\sigma P_\sigma. \tag{4.40}$$

The ideal MHD equations are the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{4.41a}$$

the momentum equation

$$\rho \frac{d\boldsymbol{u}}{dt} = \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\nabla} P, \tag{4.41b}$$

Ohm's law

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0, \tag{4.41c}$$

and the energy equation

$$\frac{d}{dt}\Big(\frac{P}{\rho^\gamma}\Big) = 0. \tag{4.41d}$$

As with the generalized MHD equations, the convective derivative operator $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$. Maxwell's equations in ideal MHD are the same as in the generalized MHD equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.42a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \tag{4.42b}$$

$$\mu_0 \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}. \tag{4.42c}$$

4.3.1 Assumptions in ideal MHD

The ideal MHD equations are derived by ignoring terms in the generalized MHD equations. Setting these terms to zero is justified by four assumptions: high collisionality, *very* high collisionality, small gyro-radius, and low resistivity. Physically, these assumptions imply that the plasma has a strong magnetic field, that changes in the plasma happen slowly in both space and time, and that collisions happen frequently but not so frequently that the electric resistivity term becomes significant.

These assumptions are consistent with the assumptions of Braginskii [2], who derived Braginskii coefficients for Π_σ, $\boldsymbol{\nabla} \cdot \boldsymbol{Q}_\sigma$, and η assuming a magnetized, collisional plasma. In section 4.3.2, we'll use these Braginskii coefficients to justify why certain terms in the generalized MHD equations can be considered negligible.

Strictly speaking, ideal MHD is only valid if the four conditions (that we discuss in more detail below) are satisfied. In fusion plasmas, the high collisionality and very high collisionality assumptions are never satisfied, due to the high temperatures and the $\nu \sim v^{-3} \sim T^{-3/2}$ dependence of the collision frequency. Nevertheless, ideal MHD is often used to study and predict the behavior of fusion plasmas, especially when solving for static MHD equilibrium. Readers interested in better understanding *why* ideal MHD is often used to study a regime where it isn't valid can refer to chapters 1 and 2 of [4].

High collisionality

Deriving ideal MHD from the generalized MHD equations requires two assumptions about collisionality. These assumptions are known as the high collisionality assumption and the very high collisionality assumption. Collisions tend to increase entropy, driving f_σ towards an isotropic Maxwellian distribution allowing for the anisotropic components of the pressure tensor Π_e and Π_i to be neglected.

The high collisionality assumption is given by

$$\frac{v_{th,i}\tau_{ii}}{L} \ll 1 \tag{4.43}$$

where $v_{th,i}$ is the ion thermal velocity, τ_{ii} is the ion collision timescale, and L is the characteristic length scale in the plasma.

The very high collisionality assumption is more stringent than the high collisionality assumption by a factor of $\sqrt{m_i/m_e} \sim 40$:

$$\left(\frac{m_i}{m_e}\right)^{1/2} \frac{v_{th,i}\tau_{ii}}{L} \ll 1. \tag{4.44}$$

The high collisionality assumption is derived from the assumption that

$$\omega\tau_i \ll 1$$

where ω is the characteristic frequency of disturbances in the MHD model and τ_i is the characteristic timescale of ion collisions. Physically, this means that changes in the plasma happen slowly. As we know from table 1 in section 1, for ions scattering collisions are dominated by collisions with other ions with a timescale τ_{ii}. Setting $\tau_i \sim \tau_{ii}$ and

$$\omega \sim \frac{v_{th,i}}{L}$$

using $\frac{\omega}{k} \sim u_{ph} \sim |\boldsymbol{u}| \sim v_{th,i}$ where u_{ph} is the phase velocity of any disturbances, this gives eq. (4.43). We set $|\boldsymbol{u}| \sim v_{th,i}$ because, while $\boldsymbol{u}$ is unknown, we know that $\boldsymbol{u} \approx \boldsymbol{u}_i$ and we can expect that disturbances in the ions will not travel faster than the ion thermal velocity.

The very high collisionality assumption eq. (4.44) does not have a clear physical origin, but it implies that changes in the plasma happen *very* slowly, by a factor of $\sqrt{m_i/m_e} \sim 40$ slower than from the high collisionality assumption.

Equation (4.43) implies that the mean free path for both ions and electrons is much less than the length scale in the plasma,

$$\frac{\lambda_{mfp,i}}{L} \sim \frac{\lambda_{mfp,e}}{L} \ll 1.$$

We can derive this from eq. (4.43) using $\lambda_{mfp,i} \sim v_{th,i}\tau_{ii}$ and $\lambda_{mfp,e} \sim \tau_{ee}v_{th,e} \sim \tau_{ii}v_{th,i}$ where factors of $\sqrt{m_i/m_e}$ have cancelled.

Small gyro-radius

The third assumption required to derive ideal MHD is the small gyro-radius assumption. This is given by

$$\frac{\rho_i}{L} \ll 1 \tag{4.45}$$

where ρ_i is the ion gyro-radius. Since the electron gyro-radius ρ_e is a factor of $\sqrt{m_e/m_i}$ smaller than the ion gyro-radius at constant temperature, than the small electron gyro-radius assumption is automatically satisfied if eq. (4.45) is satisfied.

Equation (4.45) is well satisfied in fusion experiments, as the magnetic field is typically quite strong and the experiments are typically large.

Equation (4.45) implies that the frequencies in MHD are slow relative to the ion cyclotron frequency, as shown by

$$\frac{\rho_i}{L} = \frac{mv_{th,i}}{eBL} = \frac{v_{th,i}}{\Omega_i L} = \frac{\omega}{\Omega_i} \ll 1.$$

Low resistivity

The fourth assumption required to derive ideal MHD is given by

$$\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{\rho_i}{L}\right)^2 \left(\frac{v_{th,i}\tau_{ii}}{L}\right)^{-1} \ll 1. \tag{4.46}$$

Physically, this implies that the ion gyro–radius is small and collisions are infrequent; to simultaneously satisfy the very high collisionality assumption requires that collisions are frequent but not too frequent.

Equation (4.46) can be derived from the condition that the resistivity η in the plasma is very low, so that

$$\frac{|\eta \boldsymbol{J}|}{|\boldsymbol{u} \times \boldsymbol{B}|} \ll 1. \tag{4.47}$$

Using $\eta \approx \frac{m_e \nu_{ei}}{ne^2} \approx \frac{m_e}{ne^2 \tau_{ei}}$, this can be written as

$$\frac{m_e}{ne^2 \tau_{ei}} \frac{|\boldsymbol{J}|}{v_{th,i} B} \ll 1.$$

From momentum balance, $|\boldsymbol{J}| \sim |\boldsymbol{\nabla} P_i|/|\boldsymbol{B}|$. From $P_i \sim n m_i v_{th,i}^2$, we get

$$|\boldsymbol{J}| \sim \frac{m_i n v_{th,i}^2}{LB}.$$

We also have that $\tau_{ei} \sim \tau_{ee} \sim (\frac{m_e}{m_i})^{\frac{1}{2}} \tau_{ii}$. So the inequality becomes

$$\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{m_e}{e^2 \tau_{ii}} \frac{m_i v_{th,i}}{LB^2} = \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{m_i v_{th,i}}{eBL}\right)^2 \left(\frac{L}{v_{th,i} \tau_{ii}}\right) \ll 1$$

which is identical to eq. (4.46).

4.3.2 Derivation of ideal MHD

We now derive each of the ideal MHD equations, by showing that terms in the generalized MHD equations are negligible under the assumptions of high collisionality, very high collisionality, small gyro–radius, and low resistivity.

Derivation of ideal MHD continuity equation

The ideal MHD continuity equation eq. (4.41a) is identical to the generalized MHD continuity equation eq. (4.27a), so no further approximations are required.

Derivation of ideal MHD momentum equation

To derive the ideal MHD momentum equation, we start with the generalized MHD momentum equation (eq. (4.27b)) and decomposing $\boldsymbol{P}_i$ and $\boldsymbol{P}_e$ into isotropic and anisotropic components, such that $\boldsymbol{P}_\sigma = P_\sigma \boldsymbol{I} + \boldsymbol{\Pi}_\sigma$. Using $P = P_i + P_e$, this gives

$$\rho \frac{d\boldsymbol{u}}{dt} - \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{\nabla} P = -\boldsymbol{\nabla} \cdot (\boldsymbol{\Pi}_i + \boldsymbol{\Pi}_e). \tag{4.48}$$

We will now show that the terms on the RHS are negligible relative to the terms on the LHS. By ignoring the RHS terms, the terms that remain in eq. (4.48) will be identical to the ideal MHD momentum equation (eq. (4.41b)).

From Braginskii [2], the leading-order effect on the matrix elements of $\boldsymbol{\Pi}$ is ion viscosity (the electron viscosity is smaller by a factor of $\sqrt{m_e/m_i}$), which Braginskii calculates to be

$$\Pi_{i,jj} \sim \mu\Big(2\boldsymbol{\nabla}_\parallel \cdot \boldsymbol{v}_\parallel - \frac{2}{3} \boldsymbol{\nabla} \cdot \boldsymbol{v}\Big) \sim \mu \frac{v_{th,i}}{L} \tag{4.49}$$

where $\mu \sim n T_i \tau_{ii} \sim n m_i v_{th,i}^2 \tau_{ii}$. Using $P_i \sim n k_B T_i \sim n m_i v_{th,i}^2$, we have that

$$\frac{|\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i|}{|\boldsymbol{\nabla} P_i|} \sim \mu \frac{v_{th,i}}{L^2} \frac{L}{n m_i v_{th,i}^2} \sim \left(\frac{\tau_{ii} v_{th,i}}{L}\right) \ll 1$$

due to the high collisionality assumption. Because the ion viscosity is larger than the electron viscosity, $|\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i| \gg |\boldsymbol{\nabla} \cdot \Pi_e|$, and $|\boldsymbol{\nabla} P_i \gg \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_i|$, then both RHS terms are negligible.

Derivation of ideal MHD Ohm's law

To derive the ideal MHD Ohm's law, we start with the generalized MHD Ohm's law eq. (4.27c) and rearrange terms to get

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{J} + \frac{1}{en} \boldsymbol{J} \times \boldsymbol{B} - \frac{1}{en} \boldsymbol{\nabla} P_e - \frac{1}{en} \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_e. \tag{4.50}$$

We will now show that the terms of the RHS are negligible relative to the terms on the LHS. By ignoring the RHS terms, the terms that remain in eq. (4.50) will be identical to the ideal MHD Ohm's law (eq. (4.41c)).

The first term on the RHS is negligible due to the low resistivity assumption, as we showed in eq. (4.47).

The second and third terms on the RHS are of the same magnitude, as can be shown from the ideal MHD momentum equation $\boldsymbol{J} \times \boldsymbol{B} \sim \boldsymbol{\nabla} P_e$. Thus, if the second term is negligible, the third is as well. The second term on the RHS is negligible from the small gyro-radius assumption,

$$\frac{|\boldsymbol{\nabla} P_e|}{|en\boldsymbol{u} \times \boldsymbol{B}|} \sim \frac{m_i v_{th,i}^2}{ev_{th,i}BL} \sim \frac{\rho_i}{L} \ll 1,$$

which implies the third term on the RHS is negligible as well.

The fourth term on the RHS can be shown to be negligible using the same Braginskii-based arguments as in the derivation of the ideal MHD momentum equation:

$$\frac{|\boldsymbol{\nabla} \cdot \boldsymbol{\Pi_e}|}{|\boldsymbol{\nabla} P_e|} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{\tau_{ii} v_{th,i}}{L}\right) \ll 1.$$

Thus, all four of the RHS terms in eq. (4.50) are negligible relative to the LHS terms.

Derivation of ideal MHD energy equation

To derive the ideal MHD energy equation, we start with the generalized MHD energy equations eqs. (4.27d) and (4.27e), reproduced below:

$$\frac{d}{dt}\Big(\frac{P_i}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[-\boldsymbol{\Pi}_i : \boldsymbol{\nabla} \boldsymbol{u} + \Big(\frac{\partial W_i}{\partial t}\Big)_e - \boldsymbol{\nabla} \cdot \boldsymbol{Q}_i\Big] \tag{4.51}$$

$$\frac{d}{dt}\Big(\frac{P_e}{\rho^\gamma}\Big) = \frac{2}{3\rho^\gamma}\Big[-\boldsymbol{\Pi}_e : \boldsymbol{\nabla}(\boldsymbol{u} - \frac{\boldsymbol{J}}{en_e})) + \frac{1}{en_e}\boldsymbol{J} \cdot \boldsymbol{\nabla}\Big(\frac{P_e}{\rho^\gamma}\Big) + \Big(\frac{\partial W_e}{\partial t}\Big)_i - \boldsymbol{\nabla} \cdot \boldsymbol{Q_e}\Big]. \tag{4.52}$$

We now show that each of the terms on the RHS is negligible relative to the terms on the LHS.

The first term on the RHS of eq. (4.51) is negligible relative to $\frac{\partial P_i}{\partial t}$, as we now show. Recall that, from Braginskii, the leading order contribution to Π_i is viscosity which goes like $\mu v_{th,i}/L \sim nm_i v_{th,i}^3 \tau_{ii}/L$. We also have that $\frac{\partial P_i}{\partial t} \sim \omega n m_i v_{th,i}^2$, and $\omega \sim v_{th,i}/L$. This gives

$$\frac{|\boldsymbol{\Pi}_i : \boldsymbol{\nabla} \boldsymbol{u}|}{|\frac{\partial P_i}{\partial t}|} \sim \frac{nm_i v_{th,i}^3 \tau_{ii}}{L} \frac{v_{th,i}}{L} \frac{1}{\omega m_i n v_{th,i}^2} \sim \frac{\tau_{ii} v_{th,i}}{L} \ll 1 \tag{4.53}$$

by the high collisionality assumption.

The first term on the RHS of eq. (4.52) is negligible relative to $\frac{\partial P_e}{\partial t}$, as we now show. Because the electron viscosity is smaller than the ion viscosity by a factor of $\sqrt{m_e/m_i}$, then (comparing with eq. (4.53)) we have

$$\frac{|\boldsymbol{\Pi}_e : \boldsymbol{\nabla} \boldsymbol{u}|}{|\frac{\partial P_e}{\partial t}|} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \frac{\tau_{ii} v_{th,i}}{L} \ll 1.$$

Using $|\boldsymbol{J}| \sim |\boldsymbol{\nabla} P|/|\boldsymbol{B}| \sim P/LB$, then

$$\frac{|\boldsymbol{\Pi}_e : \boldsymbol{\nabla}(\boldsymbol{J}/en)|}{|\frac{\partial P_e}{\partial t}|} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \frac{nm_i v_{th,i}^3 \tau_{ii}}{L} \frac{P}{enL^2B} \frac{L}{v_{th,i}P} \sim$$

$$\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{\tau_{ii} v_{th,i}}{L}\right) \left(\frac{m_i v_{th,i}}{eBL}\right) \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{\tau_{ii} v_{th,i}}{L}\right) \left(\frac{\rho_i}{L}\right) \ll 1.$$

Each term in parentheses is much less than 1, due to the high collisionality and small gyro-radius assumptions.

The second term on the RHS of eq. (4.52) is negligible due to the small gyro-radius assumption:

$$\frac{|(\boldsymbol{J} \cdot \boldsymbol{\nabla} P_e)/en|}{|\frac{\partial P_e}{\partial t}|} \sim \frac{m_i n v_{th,i}^2}{L^2 Ben} \frac{1}{\omega} \sim \left(\frac{\rho_i}{L}\right) \ll 1.$$

The $(\frac{\partial W_\sigma}{\partial t})_\alpha$ terms in eqs. (4.51) and (4.52) are also negligible, due to the very high collisionality assumption. Recall that $(\frac{\partial W_\sigma}{\partial t})_\alpha$ is the rate of thermal energy transfer per volume to species σ due to collisions with species α. For a two-species plasma, if $T_e = T_i$ then no energy is transferred. The timescale over which energy equilibrates is τ_{eq}, the energy equilibration time. Recall from table 1 that the energy equilibration time is larger than τ_{ii} by a factor $(\frac{m_i}{m_e})^{\frac{1}{2}}$. Based on these physical arguments, we would expect to be able to write $(\frac{\partial W_\sigma}{\partial t})_\alpha$ as

$$\left(\frac{\partial W_\sigma}{\partial t}\right)_\alpha \sim \frac{n(T_\alpha - T_\sigma)}{\tau_{eq}}.$$

This term is not small unless $T_i \approx T_e$.[19] This will occur if the energy equilibrium time is short relative to the characteristic timescale over which the plasma varies $\tau \sim \frac{1}{\omega}$. Mathematically, we can write this as $\tau_{eq}\omega \ll 1$, to ensure that $T_e \approx T_i$. But since $\omega \sim v_{th,i}/a$ and $\tau_{eq} \sim (\frac{m_i}{m_e})^{\frac{1}{2}}\tau_{ii}$, then we can write this requirement as

$$\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{v_{th,i}\tau_{ii}}{L}\right) \ll 1$$

which is much less than 1 due to the very high collisionality assumption of ideal MHD.

By neglecting all but the last term, eqs. (4.51) and (4.52) now become

$$\frac{d}{dt}\left(\frac{P_i}{\rho^\gamma}\right) = -\frac{2}{3\rho^\gamma}\Big[\boldsymbol{\nabla}\cdot\boldsymbol{Q}_i\Big]$$

$$\frac{d}{dt}\left(\frac{P_e}{\rho^\gamma}\right) = -\frac{2}{3\rho^\gamma}\Big[\boldsymbol{\nabla}\cdot\boldsymbol{Q}_e\Big].$$

Adding these equations together and using $P = P_e + P_i$, we have

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = -\frac{2}{3\rho^\gamma}\Big[\boldsymbol{\nabla}\cdot(\boldsymbol{Q}_i + \boldsymbol{Q}_e)\Big]. \tag{4.54}$$

Braginskii [2] shows that for a collisional, magnetized plasma, the heat flux $\boldsymbol{Q}_\sigma$ is proportional to the temperature gradient, and that (due to the confinement of particles perpendicular to the magnetic field) the heat flux is strongest parallel to the magnetic field. Thus,

$$Q_{\sigma\|} \approx -\kappa_\| \boldsymbol{\nabla}_\| T$$

and the energy equation becomes

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = -\frac{2}{3\rho^\gamma}\Big[\boldsymbol{\nabla}_\|\Big((\kappa_{\|,i} + \kappa_{\|,e})\boldsymbol{\nabla}_\| T\Big)\Big]$$

Braginskii also shows that

$$\frac{\kappa_{\|,i}}{\kappa_{\|,e}} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}$$

and that

$$\kappa_{\|,e} \sim \frac{nT_e\tau_{ee}}{m_e}.$$

Thus

$$\frac{\boldsymbol{\nabla}_\|(\kappa_{\|,e}\boldsymbol{\nabla}_\| T)}{\frac{\partial P}{\partial t}} \sim \frac{1}{L^2}\frac{nT^2\tau_{ee}}{m_e}\frac{1}{\omega nT} \sim \frac{m_i v_{th,i}^2}{m_e L^2 \omega}\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\tau_{ii} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{v_{th,i}\tau_{ii}}{L}\right) \ll 1$$

due to the very high collisionality assumption. Thus, the heat flux terms in the energy equation are negligible in ideal MHD. Thus, eq. (4.54) simplifies to the ideal MHD energy equation, eq. (4.41d).

4.3.3 The electric field in ideal MHD

In ideal MHD, the electric field $\boldsymbol{E}$ is not an independent variable determined by Maxwell's equations, but rather a variable dependent on $\boldsymbol{u}$ and $\boldsymbol{B}$ through eq. (4.41c). This gives

$$\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B}.$$

As a result, $E_\| = 0$, and only $\boldsymbol{E}_\perp$ contributes to $\boldsymbol{E}$.

[19] You can convince yourself of this by comparing this term with the $\frac{\partial P_\sigma}{\partial t}$ term, and setting $T_\alpha \approx 0$.

Note that, even though $\rho_e \to 0$, $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{B}) \neq 0$. This is because ideal MHD calculates the curl-free electrostatic component of $\boldsymbol{E}$ incorrectly. However, this does not affect the accuracy of the ideal MHD model, because only the divergence-free electromagnetic component of $\boldsymbol{E}$ is used to calculate $\frac{\partial \boldsymbol{B}}{\partial t}$ using Faraday's law.

$\boldsymbol{E}$ can be eliminated entirely as a dynamical variable by combining Ohm's law (eq. (4.41c)) with Faraday's law (eq. (4.42b)) to get

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t}. \tag{4.55}$$

In practice, we can forget about solving for $\boldsymbol{E}$ in ideal MHD and instead focus only on solving for ρ, $\boldsymbol{u}$, $\boldsymbol{J}$, P, and $\boldsymbol{B}$.

5 Waves in plasmas

Then (Fermi) said "Of course such waves could exist." Fermi had such authority that if he said "of course" today, every physicist said "of course" tomorrow.

HANNES ALFVÉN, ON MHD WAVES

Waves are one of the most important topic in plasma physics. One of the core courses in the Princeton plasma physics curriculum, Plasma Waves and Instabilities (AST553), is dedicated almost entirely to the study of plasma waves.

There are many types of waves in plasmas. In section 1, we derived the most basic type of wave in a plasma, plasma oscillations. These were derived by assuming stationary ions, zero magnetic field, zero temperature, and using a fluid model for the electrons. By linearizing the fluid equations and using some algebra, we obtained a characteristic frequency of $\omega_P^2 = \frac{e^2 n_0}{\epsilon_0 m_e}$. We now look at a variety of different plasma waves, including plasma oscillations in plasmas with non-zero temperature.

Section overview

In section 5.1 we introduce and derive three types of waves – Langmuir waves, ion acoustic waves, and electromagnetic plasma waves – which occur in warm (i.e., positive-temperature) unmagnetized plasmas. These waves also occur in magnetized plasmas parallel to the direction of the magnetic field. As we'll see, Langmuir waves (section 5.1.1) are plasma oscillations with finite temperature, ion acoustic waves are analogous to sound (acoustic) waves in gases, and electromagnetic plasma waves are light waves modified by the plasma. In section 5.2 we introduce and derive three types of waves – the Alfvén wave, the fast wave, and the slow wave – which arise in plasmas with a uniform background magnetic field; these can be derived using the ideal MHD equations. In section 5.3, we'll then discuss the streaming instability which arises when two plasma species have different net velocities.

5.1 Waves in unmagnetized plasmas

In this subsection, we'll look for waves that occur in a uniform warm plasma without background $\boldsymbol{E}$ or $\boldsymbol{B}$ fields. To do so, we'll use a technique called *linearization*, which finds equations that describe small perturbations to a background state. We look for wave solutions that propagate in space and time with an $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ dependence. Our goal will be to derive a *dispersion relation*, which is an equation relating the frequency $\omega = 2\pi/T$ to the wavenumber $k = 2\pi/\lambda$.

The first two waves – the Langmuir wave and the ion acoustic wave – are *electrostatic* waves. To find these waves, we assume $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, and use Gauss's law

$$-\boldsymbol{\nabla}^2\phi = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_\sigma. \tag{5.1}$$

Deriving the dispersion relation from Gauss's law requires linearizing the multi-fluid equations and calculating a *response function* that relates n_σ to ϕ.

The third wave is called the *electromagnetic* wave. To find this wave, we use the full Maxwell's equations. Deriving the dispersion relation requires linearizing Maxwell's equations and the multi-fluid equations.

Linearization

Linearization is a method of solving for the dynamics of small perturbations to a background state. Each variable is expanded around the background solution in a power series in the small quantity ϵ; the variable $n_\sigma(\boldsymbol{r}, t)$, for example, would be expanded around the background state $n_{\sigma 0}$ as

$$n_\sigma(\boldsymbol{r}, t) = n_{\sigma 0}(\boldsymbol{r}, t) + \epsilon n_{\sigma 1}(\boldsymbol{r}, t) + \frac{1}{2}\epsilon^2 n_{\sigma 2}(\boldsymbol{r}, t) + \dots$$

The zeroth-order terms (in this example, $n_{\sigma 0}$) by definition satisfy the relevant equations. The power series expansion is plugged into the relevant equations, and terms of order $\mathcal{O}(\epsilon^2)$ and smaller are ignored. Since we are looking for wave solutions, we then assume that the first-order terms (proportional to ϵ^1) have exponential dependence $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ and look for solutions to the first-order equations.

We now show how the method of linearization can be used to solve for electrostatic response functions that relate the first-order density perturbation $n_{\sigma 1}$ to the first-order electric potential perturbation ϕ_1, using both the kinetic Vlasov-

Maxwell model as well as the multi-fluid model. These response functions will be used to derive dispersion relations for electrostatic wave in sections 5.1.1 and 5.1.2.

Kinetic response function

We now use a kinetic model and the method of linearization to derive the response function relating the first-order density perturbation $n_{\sigma 1}$ to the first-order electric potential perturbation ϕ_1. Our starting point is Vlasov-Maxwell equation, eq. (3.19). We assume a stationary uniform background plasma with no background $\boldsymbol{E}$ or $\boldsymbol{B}$ fields and no collisions, so that

$$f_\sigma(\boldsymbol{r}, \boldsymbol{v}, t) = f_{\sigma 0}(\boldsymbol{v}) + \epsilon f_{\sigma 1}(\boldsymbol{r}, \boldsymbol{v}, t) + \mathcal{O}(\epsilon)^2$$
$$\phi(\boldsymbol{r}, t) = \epsilon \phi_1(\boldsymbol{r}, t) + \mathcal{O}(\epsilon^2).$$

We have assumed that the electromagnetic field is electrostatic, so that $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ and the first-order perturbation $\boldsymbol{B}_1$ is negligible. Linearizing the Vlasov-Maxwell equation and keeping terms first-order in ϵ gives

$$\frac{\partial f_{\sigma 1}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_{\sigma 1} - \frac{q_\sigma}{m_\sigma} \boldsymbol{\nabla}\phi_1 \cdot \boldsymbol{\nabla}_v f_{\sigma 0} = 0. \tag{5.2}$$

Assuming an exponential dependence $e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ for each of the first-order quantities, this becomes

$$-i\omega f_{\sigma 1} + \boldsymbol{v} \cdot i\boldsymbol{k} f_{\sigma 1} - \frac{q_\sigma}{m_\sigma} i\phi_1 \boldsymbol{k} \cdot \boldsymbol{\nabla}_v f_{\sigma 0} = 0.$$

Solving for $f_{\sigma 1}$ gives

$$f_{\sigma 1}(k, \omega) = \frac{q_\sigma \phi_1}{m_\sigma} \frac{\frac{\partial f_{\sigma 0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}}$$

where $v_\parallel$ is parallel to $\boldsymbol{k}$. The first-order density perturbation can be found by integrating $f_{\sigma 1}$ over velocity space:

$$n_{\sigma 1}(k, \omega) = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial f_{\sigma 0}}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} d^3\boldsymbol{v} = \frac{q_\sigma \phi_1}{m_\sigma} \int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} dv_\parallel \tag{5.3}$$

where $g_\sigma = \int f_{\sigma 0} d^2\boldsymbol{v}_\perp$. This integral cannot be performed naively, due to the $v_\parallel - \omega/k$ term in the denominator which blows up when $v_\parallel = \omega/k$. A proper treatment requires a Laplace transform rather than a Fourier transform and results in Landau damping. Instead, we'll use a non-rigorous treatment and look at two limits of eq. (5.3): the adiabatic limit, when $\omega/k \gg v_{th,\sigma}$, and the isothermal limit, when $\omega/k \ll v_{th,\sigma}$. The adiabatic limit is the fast wave limit, when the wave moves much faster than the particles. The isothermal limit is the slow wave limit, when the particles move much faster than the wave.

To derive the kinetic response function for species σ in the adiabatic limit, we first integrate by parts:

$$\int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}} dv_\parallel = \int \frac{g_\sigma}{(v_\parallel - \frac{\omega}{k})^2} dv_\parallel + \Big[\frac{g_\sigma}{v_\parallel - \frac{\omega}{k}}\Big]_{-\infty}^{\infty}.$$

The second term on the RHS is zero because f_σ (as well as g_σ) is zero at $v_\parallel \to \pm\infty$. Next, we write

$$\int \frac{g_\sigma}{(v_\parallel - \frac{\omega}{k})^2} dv_\parallel = \frac{k^2}{\omega^2} \int \frac{g_\sigma}{(1 - \frac{kv_\parallel}{\omega})^2} dv_\parallel .$$

In the adiabatic limit where $v_{th,\sigma}k/\omega \ll 1$, then g_σ is approximately zero except in the regime $v_\parallel \sim \pm v_{th,\sigma}$. Thus, in the adiabatic limit $v_\parallel k/\omega \ll 1$ in the non-zero portion of the above integral as well.Thus, using the small-x expansion

$$(1 + ax)^b = 1 + abx + a^2(b)(b-1)\frac{x^2}{2!} + ...$$

the response function becomes

$$n_{\sigma 1} \approx \frac{q_\sigma \phi_1}{m_\sigma} \frac{k^2}{\omega^2} \int g_\sigma \Big(1 + 2\frac{kv_\parallel}{\omega} + 3\frac{k^2 v_\parallel^2}{\omega^2}\Big) dv_\parallel .$$

The first term in the above expression integrates to $n_{\sigma 0}$, the second term integrates to zero (assuming the mean velocity in the parallel direction is zero), and the third term cannot be calculated exactly but scales as $\gamma v_{th,\sigma}^2 n_{\sigma 0} k^2/\omega^2$ where γ is an unknown constant. Thus, the kinetic response function in the adiabatic limit is

$$n_{\sigma 1}(k, \omega) = \frac{q_\sigma n_{\sigma 0} k^2}{m_\sigma \omega^2} \Big(1 + \gamma \frac{k^2 v_{\text{th}\sigma}^2}{\omega^2}\Big)\phi_1. \tag{5.4}$$

To derive the kinetic response function for species σ in the isothermal limit, we assume a Maxwellian distribution for $f_{\sigma 0}$

$$f_{\sigma 0} = n_{\sigma 0}\Big(\frac{m_\sigma}{2\pi k_B T_\sigma}\Big)^{3/2} \exp\Big(-\frac{m_\sigma v^2}{2k_B T_\sigma}\Big).$$

Integrating over the perpendicular directions gives

$$g_\sigma = n_{\sigma 0}\Big(\frac{m_\sigma}{2\pi k_B T_\sigma}\Big)^{1/2} \exp\Big(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}\Big).$$

Taking the derivative,

$$\frac{\partial g_\sigma}{\partial v_\parallel} = -n_{\sigma 0}\frac{m_\sigma^{3/2} v_\parallel}{(2\pi)^{1/2}(k_B T_\sigma)^{3/2}} \exp(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}).$$

Solving for the response function requires performing the integral in eq. (5.3). We plug in our expression for $\frac{\partial g_\sigma}{\partial v_\parallel}$ and make the approximation that, in the isothermal limit,

$$n_{\sigma 1} = \frac{q_\sigma \phi_1}{m_\sigma}\int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel - \frac{\omega}{k}}\, dv_\parallel \approx \frac{q_\sigma \phi_1}{m_\sigma}\int \frac{\frac{\partial g_\sigma}{\partial v_\parallel}}{v_\parallel}\, dv_\parallel\,.$$

We argue this approximation is justified because, since $\omega/k \ll v_{th,\sigma}$, then $v_\parallel = \omega/k$ near the top of the Maxwellian distribution when $\frac{\partial g_\sigma}{\partial v_\parallel} \approx 0$. This means that the numerator is approximately zero when the denominator blows up, so the blowup of the denominator can (non-rigorously) be ignored. Making this approximation, the $v_\parallel$ terms in the numerator and denominator cancel and we are left with

$$n_{\sigma 1} = -\frac{q_\sigma m_\sigma^{1/2} n_{\sigma 0}\phi_1}{(k_B T_\sigma)^{3/2}(2\pi)^{1/2}}\int \exp\Big(-\frac{m_\sigma v_\parallel^2}{2k_B T_\sigma}\Big)\, dv_\parallel\,.$$

Using $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, the integral becomes $\sqrt{\frac{2\pi k_B T_\sigma}{m_\sigma}}$. We also have $k_B T_\sigma/m_\sigma = v_{th,\sigma}^2$. Thus, the kinetic response function in the isothermal limit is

$$n_{\sigma 1}(k,\omega) = -\frac{q_\sigma n_{\sigma 0}}{m_\sigma v_{\text{th}\sigma}^2}\phi_1. \tag{5.5}$$

Fluid response function

We now use the multi-fluid model and the method of linearization to derive the response function relative the first-order density perturbation $n_{\sigma 1}$ to the first-order electric potential perturbation ϕ_1. Our starting point is the multi-fluid continuity equation eq. (4.1a) and multi-fluid momentum equation eq. (4.1b). We assume a collisionless plasma ($\boldsymbol{R}_{\sigma\alpha}$) and electrostatic perturbations ($\boldsymbol{B} = 0$, $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$). We assume a stationary background plasma with constant density and scalar pressure. Linearizing around this background, we have

$$n_\sigma(\boldsymbol{r},t) = n_{\sigma 0} + \epsilon n_{\sigma 1}(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)$$
$$\boldsymbol{u}_\sigma(\boldsymbol{r},t) = \epsilon \boldsymbol{u}_{\sigma 1}(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)$$
$$P_\sigma(\boldsymbol{r},t) = P_{\sigma 0} + \epsilon P_{\sigma 1}(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)$$
$$\phi(\boldsymbol{r},t) = \epsilon\phi_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2).$$

Linearizing the multi-fluid continuity equation (eq. (4.1a)) and keeping terms first-order in ϵ gives

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0}\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\sigma 1} = 0.$$

Linearizing the multi-fluid momentum equation (eq. (4.1b)) and keeping terms first-order in ϵ gives

$$\frac{\partial \boldsymbol{u}_{\sigma 1}}{\partial t} = -\frac{q_\sigma}{m_\sigma}\boldsymbol{\nabla}\phi_1 - \frac{1}{m_\sigma n_{\sigma 0}}\boldsymbol{\nabla}P_{\sigma 1}.$$

Assuming an exponential dependence $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ for each of the first-order quantities, these become

$$-i\omega n_{\sigma 1} + i n_{\sigma 0}\boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 1} = 0$$

$$-i\omega \boldsymbol{u}_{\sigma 1} = -i\frac{q_\sigma}{m_\sigma}\boldsymbol{k}\phi_1 - \frac{iP_{\sigma 1}\boldsymbol{k}}{m_\sigma n_{\sigma 0}}.$$

Solving for $n_{\sigma 1}$ gives

$$n_{\sigma 1} = \frac{n_{\sigma 0}\boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 1}}{\omega} = \frac{k^2}{\omega^2}\Big(\frac{n_{\sigma 0}q_\sigma}{m_\sigma}\phi_1 + \frac{P_{\sigma 1}}{m_\sigma}\Big). \tag{5.6}$$

Solving for the response function requires eliminating $P_{\sigma 1}$, which can be done using the multi-fluid energy equation eq. (4.1c). We consider two limiting cases, the adiabatic limit where $\omega/k \gg v_{th,\sigma}$, and the isothermal limit where $\omega/k \ll v_{th,\sigma}$. In the adiabatic limit, changes to the plasma due to the wave happen quickly, so the energy equation simplifies to eq. (4.22). In the isothermal limit, changes to the plasma happen slowly, so the plasma temperature has time to equilibrate and so the energy equation simplifies to $\boldsymbol{\nabla} T_\sigma = 0$. We now calculate the response function in these limits.

In the adiabatic limit, the energy equation is given by eq. (4.22). Linearizing, this becomes

$$\frac{dP_{\sigma 1}}{dt}\frac{1}{n_{\sigma 0}^{\gamma}} - \frac{\gamma P_{\sigma 0}}{n_{\sigma 0}^{\gamma+1}}\frac{dn_{\sigma 1}}{dt} = 0.$$

To first order in ϵ, $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}_{\sigma 1}\cdot\boldsymbol{\nabla} \approx \frac{\partial}{\partial t}$ since the zeroth-order terms are uniform in space. Thus, the linearized adiabatic energy equation becomes

$$\frac{\partial P_{\sigma 1}}{\partial t} = \frac{\gamma P_{\sigma 0}}{n_{\sigma 0}}\frac{\partial n_{\sigma 1}}{\partial t}$$

$$-i\omega P_{\sigma 1} = -i\omega\frac{\gamma P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1}$$

$$P_{\sigma 1} = \frac{\gamma P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1} = \gamma m_\sigma v_{th\sigma}^2 n_{\sigma 1}$$

using $P_{\sigma 0} = n_{\sigma 0}k_B T_{\sigma 0}$ and $v_{th\sigma}^2 = k_B T_\sigma/m_\sigma$. Plugging $P_{\sigma 1}$ into eq. (5.6) and solving for $n_{\sigma 1}$ gives

$$n_{\sigma 1} = \frac{n_{\sigma 0}q_\sigma}{m_\sigma}\frac{\phi_1}{\left(\frac{\omega^2}{k^2} - \gamma v_{th\sigma}^2\right)}. \tag{5.7}$$

In the adiabatic limit where $\omega/k \gg v_{th\sigma}$, we can Taylor expand eq. (5.7) to get

$$n_{\sigma 1}(k,\omega) = \frac{q_\sigma n_{\sigma 0}k^2}{m_\sigma \omega^2}\Big(1 + \gamma\frac{k^2 v_{\text{th}\sigma}^2}{\omega^2}\Big)\phi_1. \tag{5.8}$$

In the isothermal limit, the energy equation is given by eq. (4.23). Linearizing, this becomes

$$\boldsymbol{\nabla}\Big(\frac{P_\sigma}{n_\sigma}\Big) = \frac{1}{n_{\sigma 0}}\boldsymbol{\nabla} P_{\sigma 1} - \frac{P_{\sigma 0}}{n_{\sigma 0}^2}\boldsymbol{\nabla} n_{\sigma 1} = 0$$

$$i\boldsymbol{k}P_{\sigma 1} = \frac{i\boldsymbol{k}P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1}$$

$$P_{\sigma 1} = \frac{P_{\sigma 0}}{n_{\sigma 0}}n_{\sigma 1} = m_\sigma v_{th\sigma}^2 n_{\sigma 1}.$$

This is identical to the expression for $P_{\sigma 1}$ in the adiabatic limit, except with $\gamma = 1$. Thus, in the isothermal limit we can use eq. (5.7) with $\gamma = 1$. In the isothermal limit where $\omega/k \ll v_{th\sigma}$, we Taylor expand eq. (5.7) in the opposite limit and (keeping only the lowest-order term) get

$$n_{\sigma 1}(k,\omega) = -\frac{q_\sigma n_{\sigma 0}}{m_\sigma v_{th\sigma}^2}\phi_1. \tag{5.9}$$

Notice that the kinetic response functions in the adiabatic (eq. (5.4)) and isothermal (eq. (5.5)) limits are identical to the multi-fluid response functions in the adiabatic (eq. (5.8)) and isothermal (eq. (5.9)) limits. Thus, both kinetic and fluid models can be used to derive the dispersion relation of electrostatic plasma waves.

5.1.1 Langmuir wave

The first electrostatic wave we will introduce is usually called the Langmuir wave but is sometimes called the Bohm-Gross or Bohm wave. The Langmuir wave is the finite-temperature analogue of the plasma oscillation (derived in section 1). Langmuir oscillations are high-frequency oscillations, with a phase velocity much larger than the thermal velocities in the plasma. The dispersion relation for the Langmuir wave is

$$\omega^2 = \omega^2 = \sum_\sigma \omega_{p\sigma}^2 + \sum_\sigma \frac{\omega_{p\sigma}^2}{\omega_{pe}^2}\gamma k^2 v_{th\sigma}^2 \tag{5.10}$$

where $\omega_{p\sigma}$ is the plasma frequency for species σ. Because the electron plasma frequency and electron thermal velocity are both a factor of $\sqrt{m_e/m_i}$ larger than the ion plasma frequency and ion thermal velocity, the ion terms are usually neglected and only the electron terms are included in the dispersion relation:

$$\omega^2 = \omega_{pe}^2 + \gamma k^2 v_{th,e}^2. \tag{5.11}$$

Physical intuition

In section 1, we saw a clear physical picture for how and why plasma oscillations develop in cold (zero-temperature) plasmas. The Langmuir wave is a plasma oscillation for finite-temperature plasmas, where the finite-temperature effects modify the dispersion relation. Unfortunately, I don't have a clear physical intuition for how or why the finite temperature modifies the dispersion relation.

Derivation

To derive the dispersion relation for electrostatic waves requires using Gauss's law (eq. (5.1)) and the response functions derived earlier. The Langmuir wave is a fast wave, and can be derived in the regime that $\omega/k \gg v_{th\sigma}$ for both ions and electrons. Linearizing Gauss's law gives

$$-k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1}. \tag{5.12}$$

Plugging in the adiabatic response function (eq. (5.8)) gives

$$-k^2\phi_1 = \sum_\sigma \frac{q_\sigma^2 n_{\sigma 0}}{\epsilon_0 m_\sigma}\frac{k^2}{\omega^2}\left(1 + \frac{\gamma k^2 v_{th\sigma}^2}{\omega^2}\right)\phi_1 = \sum_\sigma \omega_{p\sigma}^2 \frac{k^2}{\omega^2}\left(1 + \frac{\gamma k^2 v_{th\sigma}^2}{\omega^2}\right)\phi_1$$

$$\omega^2 = \sum_\sigma \omega_{p\sigma}^2\left(1 + \frac{\gamma k^2 v_{th\sigma}^2}{\omega^2}\right). \tag{5.13}$$

This is a fourth-degree polynomial for ω; we can solve this equation perturbatively. Since $\omega/k \gg v_{th\sigma}$, then to lowest order $\omega^2 = \sum_\sigma \omega_{p\sigma}^2 \approx \omega_{pe}^2$. Plugging $\omega^2 = \omega_{pe}^2$ into the RHS of eq. (5.13) gives eq. (5.10).

5.1.2 Ion acoustic wave

The second electrostatic wave we will introduce is called the ion acoustic wave. The ion acoustic wave is a medium-frequency wave: the phase speed is faster than the ion thermal velocity but is slower than the electron thermal velocity. This is given by the relation $v_{th,i} \ll \frac{\omega}{k} \ll v_{th,e}$. The dispersion relation for the ion acoustic wave (in the limit that $T_e \gg T_i$) is

$$\omega^2 = k^2(c_s^2 + \gamma v_{th,i}^2) \tag{5.14}$$

where c_s is the plasma sound speed, given by

$$c_s^2 = \frac{k_B T_e}{m_i}.$$

Physical intuition

In a non-ionized gas, a pressure force (due to density perturbations) causes density perturbations to propagate with phase velocity $c_s = \gamma P/\rho$. The pressure P of the bulk gas provides the restoring force, and the mass ρ of the bulk gas provides the inertia.

The ion acoustic wave in a plasma is analogous to a sound wave in a non-ionized gas. The ion acoustic wave, like a sound wave, involves the propagation of density perturbations due to a restoring pressure force and inertia. The restoring force is provided by the electron pressure $P_e = n_e T_e$, while the inertia is provided by the ions $\rho \approx n_i m_i$.

Because $\omega/k \ll v_{th,e}$ in an ion acoustic wave, the electrons are isothermal and will Debye shield any ion density perturbations. The ions, by contrast, experience a force due to an electric field proportional to the electron temperature T_e. In other words, the electric field provides the force that allows for the interaction between electrons and ions in an ion acoustic wave.

Derivation

We once again use the linearized Gauss's law (eq. (5.12)) and the response functions derived earlier. The ion acoustic wave is in the regime $v_{th,i} \ll \frac{\omega}{k} \ll v_{th,e}$, so the ions use the adiabatic response function (eq. (5.8)) while the electrons use the isothermal response function (eq. (5.9)). Assuming a two-species plasma of ions and electrons, eq. (5.12) becomes

$$k^2\phi_1 = \left(\frac{q_i^2 n_{i0} k^2}{\epsilon_0 m_i \omega^2}\left(1+\gamma\frac{k^2 v_{\text{th},i}^2}{\omega^2}\right) - \frac{e^2 n_{e0}}{\epsilon_0 m_e v_{th,e}^2}\right)\phi_1$$
$$1 = \frac{\omega_{pi}^2}{\omega^2}\left(1+\gamma\frac{k^2 v_{\text{th},i}^2}{\omega^2}\right) - \frac{\omega_{pe}^2}{v_{th,e}^2 k^2}$$
$$\omega^2\left(\frac{\omega_{pe}^2}{v_{th,e}^2 k^2}+1\right) = \omega_{pi}^2\left(1+\gamma\frac{k^2 v_{\text{th},i}^2}{\omega^2}\right).$$

This is another fourth-order polynomial for ω. Using $v_{th,e}^2/\omega_{pe}^2 = \lambda_{De}^2$ and assuming the long-wavelength limit of $\lambda/\lambda_{De} = \frac{2\pi}{k\lambda_{De}} \gg 1$, then we can write

$$\frac{\omega_{pe}^2}{v_{th,e}^2 k^2} + 1 \approx \frac{\omega_{pe}^2}{v_{th,e}^2 k^2}.$$

The fourth-order polynomial can then be written as

$$\omega^4 - c_s^2 k^2 \omega^2 - \gamma k^4 v_{th,i}^2 c_s^2 = 0$$

which has the solution

$$\omega^2 = c_s^2 k^2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4\gamma\frac{T_i}{T_e}}\right).$$

In the limit that $T_i \ll T_e$, a Taylor expansion gives eq. (5.14).

5.1.3 Isothermal electrostatic waves don't propagate

The Langmuir wave is a fast wave for which both ions and electrons are in the adiabatic limit ($\omega/k \gg v_{th,e}$ and $\omega/k \gg v_{th,i}$). The ion acoustic wave is a slower wave for which ions are in the adiabatic limit but electrons are in the isothermal limit ($\omega/k \gg v_{th,e}$ and $\omega/k \gg v_{th,i}$). Naturally, we might also look for slow wave in which both electrons and ions are in the isothermal limit ($\omega/k \ll v_{th,e}$ and $\omega/k \ll v_{th,i}$). However, as we will now show, purely isothermal waves do not propagate.

As before, we plug the response function into the linearized Gauss's law (eq. (5.12)). Assuming isothermal response functions for each species σ, Gauss's law becomes

$$k^2\phi_1 = -\sum_\sigma \frac{1}{\lambda_D^2}\phi_1.$$

This equation has no dependence on ω. In fact, this is identical to eq. (1.21), an equation for Debye shielding, except without the test charge Q. Isothermal plasma oscillations are simply Debye shielded by both ions and electrons.

5.1.4 Electromagnetic plasma wave

Next we consider an electromagnetic wave propagating in a cold, uniform, unmagnetized plasma. The dispersion relation of the electromagnetic plasma wave is given by

$$\omega^2 = k^2c^2 + \sum_\sigma \omega_{p\sigma}^2 \tag{5.15}$$

where c is the speed of light. Note that this is the same as the dispersion relation for light waves $\omega^2 = k^2c^2$, except with a correction term due to the plasma.

Calculating the phase velocity

$$v_{ph} = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}} \geq c$$

we see that it is greater than the speed of light. This means that the index of refraction $n = c/v_{ph}$ in a plasma is less than 1. The group velocity, however, is not greater than the speed of light:

$$v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{c^2k^2 + \omega_p^2}} = \frac{c}{\sqrt{1 + \frac{\omega_p^2}{c^2k^2}}} \leq c.$$

Physical intuition

The electromagnetic plasma wave is simply a light wave modified by the plasma. Unfortunately, I don't have a good physical picture to explain how and why the electromagnetic dispersion relation is modified by the plasma.

The electromagnetic dispersion relation (eq. (5.15)) shows that k must be imaginary for $\omega < \omega_p$, implying that electromagnetic waves lower than the plasma frequency cannot propagate in a plasma. Physically, this happens because the plasma has enough time to Debye shield electric fields created by low-frequency light waves.

Derivation

To derive the dispersion relation for electromagnetic waves, we linearize Maxwell's equations and the multi-fluid equations around a cold uniform equilibrium. To lowest order in ϵ, the linearized momentum equation becomes

$$m_\sigma n_{\sigma 0} \frac{\partial \boldsymbol{u}_{\sigma 1}}{\partial t} = q_\sigma n_{\sigma 0} \boldsymbol{E}_1.$$

The linearized Faraday's law and Ampere's law become

$$\boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_1 = \mu_0 \boldsymbol{J}_1 + \frac{1}{c^2}\frac{\partial \boldsymbol{E}_1}{\partial t}.$$

Taking the curl of Ampere's law, using $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$, using $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and plugging in Faraday's law gives

$$-\boldsymbol{\nabla}^2 \boldsymbol{B}_1 = \mu_0 \boldsymbol{\nabla} \times \boldsymbol{J}_1 - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}_1}{\partial t^2}. \tag{5.16}$$

Using

$$\boldsymbol{J}_1 = \sum_\sigma q_\sigma n_{\sigma 0} \boldsymbol{u}_{\sigma 1},$$

we have

$$\frac{\partial}{\partial t} \boldsymbol{\nabla} \times \boldsymbol{J}_1 = \sum_\sigma q_\sigma n_{\sigma 0} \frac{\partial (\boldsymbol{\nabla} \times \boldsymbol{u}_{\sigma 1})}{\partial t} = \sum_\sigma \frac{q_\sigma^2 n_{\sigma 0}}{m_\sigma} \boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{\partial}{\partial t} \sum_\sigma \frac{q_\sigma^2 n_{\sigma 0}}{m_\sigma} \boldsymbol{B}_1.$$

Assuming an exponential dependent $e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ for all first-order quantities, this becomes

$$-i\omega \boldsymbol{\nabla} \times \boldsymbol{J}_1 = -i\omega \sum_\sigma \omega_{pe}^2 \epsilon_0 \boldsymbol{B}_1.$$

Plugging this into eq. (5.16) gives

$$k^2 \boldsymbol{B}_1 = -\frac{1}{c^2} \sum_\sigma \omega_{p\sigma}^2 \boldsymbol{B}_1 + \frac{\omega^2}{c^2} \boldsymbol{B}_1.$$

Eliminating $\boldsymbol{B}_1$ gives the dispersion relation eq. (5.15).

5.2 Waves in magnetized plasmas

We now consider waves that arise in a uniform warm plasma with a constant background field magnetic field $\boldsymbol{B}_0$. To do so, we'll linearize the ideal MHD equations to first order and assume an exponential $e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - i\omega)}$ dependence for each first-order perturbation. We will find three types of waves, called the shear Alfvén (or intermediate) wave, the fast wave, and the slow wave. Each wave involves the interaction of perturbations to the plasma with perturbations of the magnetic field. Our goal will be to derive dispersion relations for each wave.

Linearizing the ideal MHD equations gives a second-order equation of motion. Assuming an exponential dependence turns this into an eigenvalue equation for ω^2. The eigenvalues represent the dispersion relation for each wave, while the eigenvectors represent the direction of the perturbations that create each wave.

Linearized MHD equation of motion

We now linearize the MHD equations, deriving an equation of motion for linear perturbations to a uniform magnetized plasma. The linearized plasma variables in ideal MHD are

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)\\ \rho(\boldsymbol{r},t) &= \rho_0 + \epsilon \rho_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)\\ \boldsymbol{u}(\boldsymbol{r},t) &= \epsilon \boldsymbol{u}_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)\\ \boldsymbol{J}(\boldsymbol{r},t) &= \epsilon \boldsymbol{J}_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)\\ P(\boldsymbol{r},t) &= P_0 + \epsilon P_1(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Note that we can eliminate $\boldsymbol{E}$ from the ideal MHD equations using eq. (4.55). Linearizing the ideal MHD equations and dropping terms $\mathcal{O}(\epsilon^2)$ and smaller gives

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla}\cdot \boldsymbol{u}_1 &= 0\\ \rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} &= -\boldsymbol{\nabla} P_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_0\\ \frac{\partial P_1}{\partial t} - \gamma \frac{\partial \rho_1}{\partial t}\frac{P_0}{\rho_0} &= 0\\ \boldsymbol{J}_1 &= \frac{1}{\mu_0}\boldsymbol{\nabla}\times \boldsymbol{B}_1\\ \frac{\partial \boldsymbol{B}_1}{\partial t} &= \boldsymbol{\nabla}\times(\boldsymbol{u}_1 \times \boldsymbol{B}_0).\end{aligned}$$

We can eliminate ρ_1 and $\boldsymbol{J}_1$ by plugging the first and fourth equations into the second and third equations:

$$\begin{aligned}\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} &= -\boldsymbol{\nabla} P_1 + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times \boldsymbol{B}_1)\times \boldsymbol{B}_0\\ \frac{\partial P_1}{\partial t} &= -\gamma P_0 \boldsymbol{\nabla}\cdot \boldsymbol{u}_1.\end{aligned}$$

We now introduce the so-called displacement vector $\boldsymbol{\xi}$, defined by

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{u}_1.$$

From this definition, we can see that $\boldsymbol{\xi}$ is analogous to a position coordinate; it tells us how far and in what direction the plasma has become displaced from equilibrium. Eliminating $\boldsymbol{u}_1$ in favor of $\boldsymbol{\xi}$, the linearized MHD equations become

$$\begin{aligned}\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\boldsymbol{\nabla} P_1 + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times \boldsymbol{B}_1)\times \boldsymbol{B}_0\\ \frac{\partial \boldsymbol{B}_1}{\partial t} &= \frac{\partial}{\partial t}\Big(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times \boldsymbol{B}_0)\Big)\\ \frac{\partial P_1}{\partial t} &= -\frac{\partial}{\partial t}(\gamma P_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}).\end{aligned}$$

Integrating the second and third equations with respect to time gives

$$\begin{aligned}\boldsymbol{B}_1 &= \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times \boldsymbol{B}_0)\\ P_1 &= -\gamma P_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}.\end{aligned}$$

Plugging these into the the first equation gives the linearized MHD equation of motion,

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \gamma P_0 \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \frac{1}{\mu_0}\bigg(\boldsymbol{\nabla}\times\Big(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times \boldsymbol{B}_0)\Big)\bigg)\times \boldsymbol{B}_0. \tag{5.17}$$

This is an equation for $\boldsymbol{\xi}$ in terms of only zeroth-order quantities. Assuming an exponential $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ dependence of $\boldsymbol{\xi}$ gives

$$-\rho_0 \omega^2 \boldsymbol{\xi} = -\gamma P_0 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\xi}) - \frac{1}{\mu_0}\bigg(\boldsymbol{k}\times\Big(\boldsymbol{k}\times(\boldsymbol{\xi}\times \boldsymbol{B}_0)\Big)\bigg)\times \boldsymbol{B}_0. \tag{5.18}$$

This can be written as

$$\mathbf{M}(\boldsymbol{B}_0,\boldsymbol{k})\boldsymbol{\xi} = \omega^2 \boldsymbol{\xi} \tag{5.19}$$

where $\boldsymbol{M}(\boldsymbol{B}_0,\boldsymbol{k})$ is a matrix. This is an eigenvalue equation for ω^2, which we will use to derive the dispersion relations and eigenvectors of the three MHD waves.

5.2.1 Shear Alfvén wave

There are three types of waves that arise in a uniform magnetized plasma. The first wave we will consider is called the intermediate wave or the shear Alfvén wave. Its dispersion relation is given by

$$\omega^2 = \frac{(\boldsymbol{k} \cdot \boldsymbol{B_0})^2}{\mu_0 \rho_0} = v_A^2 k_{\parallel}^2 \tag{5.20}$$

where

$$v_A = \frac{B_0}{\sqrt{\rho_0 \mu_0}}$$

is the Alfven velocity and $k_{\parallel}$ is the component of $\boldsymbol{k}$ in the direction parallel to the magnetic field. The eigenvector is perpendicular to the background magnetic field, so that

$$\boldsymbol{\xi} \cdot \boldsymbol{B}_0 = 0. \tag{5.21}$$

Physical intuition

The shear Alfvén wave propagates parallel to the background magnetic field. The shear Alfvén wave involves a sinusoidal perturbation to the magnetic field in the direction perpendicular to the background magnetic field. The magnetic fields lines due to the shear Alfvén wave therefore appear sinusoidal, like a string after it is plucked.

The shear Alfvén wave can be understood as a consequence of the frozen-in flux theorem in ideal MHD, which we have not discussed in these notes. According to the frozen-in flux theorem, in ideal MHD the magnetic flux Φ_B through a loop moving with the fluid is constant. This implies that fluid particles are constrained to stay on magnetic field lines. The shear Alfvén wave arises because, when fluid particles are perturbed from the background magnetic field, the magnetic field must be perturbed to satisfy the frozen-in flux theorem. The perturbed magnetic field causes a magnetic tension force which acts as a restoring force.

The phase velocity of shear Alfvén waves, like the phase velocity of sound waves and ion acoustic waves, is given by the square root of the restoring force divided by the inertia. In shear Alfvén waves, the restoring force is provided by magnetic tension B^2/μ_0, while the inertia is given by ρ, giving a phase velocity equal to the Alfvén speed v_A.

Derivation

To derive the dispersion relation for the MHD waves, we look for solutions to the eigenvalue equation eq. (5.18). One way to do this is to find the solutions to

$$\det\big(\boldsymbol{M} - \omega^2 \boldsymbol{I}\big) = 0.$$

However, we can also look for solutions to eq. (5.18) using *a priori* knowledge of the eigenvectors. To derive the dispersion relation for the shear Alfvén wave, we look for a solution to eq. (5.18) using *a priori* knowledge.

We start by defining a coordinate system where

$$\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$$
$$\boldsymbol{k} = k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}}.$$

This coordinate system is fully general and describes any $\boldsymbol{B}_0$ and $\boldsymbol{k}$.

Next we'll assume (using *a priori* knowledge about the eigenvector of the shear Alfvén wave) that

$$\boldsymbol{\xi} = \xi_y \hat{\boldsymbol{y}}.$$

This implies that

$$\boldsymbol{k} \cdot \boldsymbol{\xi} = 0$$

which eliminates the first term on the RHS of eq. (5.18). Because $\boldsymbol{k}$, $\boldsymbol{B}_0$, and $\boldsymbol{\xi}$ are perpendicular, the second term on the RHS of eq. (5.18) simplifies to

$$-\frac{1}{\mu_0}\Big(\boldsymbol{k} \times \big(\boldsymbol{k} \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)\big)\Big) \times \boldsymbol{B}_0 = -\frac{k_\parallel^2 B_0^2}{\mu_0}\boldsymbol{\xi}.$$

Thus, eq. (5.18) becomes

$$-\rho_0 \omega^2 \boldsymbol{\xi} = -\frac{k_\parallel^2 B_0^2}{\mu_0}\boldsymbol{\xi}$$

which gives the dispersion relation eq. (5.20).

5.2.2 Fast and slow MHD waves

The second and third types of MHD waves that we will consider are called the fast and slow waves. Their dispersion relations are given by

$$\frac{\omega^2}{k^2} = \frac{(c_s^2 + v_A^2)}{2} \pm \frac{1}{2}\sqrt{(v_A^2 - c_s^2)^2 + 4v_A^2 c_s^2 \sin^2\theta} \tag{5.22}$$

where $c_s = \sqrt{\gamma P_0/\rho_0}$ is the sound speed and θ is the angle between the background magnetic field $\boldsymbol{B}_0$ and the direction of propagation $\boldsymbol{k}$. The fast wave corresponds to the plus solution, while the slow wave corresponds to the minus solution.

Physical intuition

As we can see from eq. (5.22), the dispersion relation for the fast and slow waves depends on the angle θ between the background magnetic field and the direction of propagation. For arbitrary θ, the dispersion relation does not have a simple intuitive algebraic form. However, the special cases $\theta = 0$ and $\theta = \pi/2$ do have simple expressions and physically intuitive meanings.

$\boldsymbol{\theta}$ = 0: When $\theta = 0$, then $\boldsymbol{k} \parallel \boldsymbol{B}_0$. In this case, eq. (5.22) simplifies to

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + v_A^2}{2} \pm \frac{|c_s^2 - v_A^2|}{2}$$

which has the two solutions

$$\frac{\omega^2}{k^2} = v_A^2, \text{ and } \frac{\omega^2}{k^2} = c_s^2.$$

Which solution corresponds to the 'fast' wave and which corresponds to the 'slow' wave depends on whether the Alfvén velocity or the sounds speed is larger. However, the physical meaning of each mode for the case $\theta = 0$ is clear. The $\omega^2 = k^2 c_s^2$ mode is a sound wave, propagating parallel to the magnetic field. This is equivalent to the ion acoustic wave in unmagnetized plasmas, except using the ideal MHD approximation instead of the multi-fluid approximation. The $\omega^2 = k^2 v_A^2$ mode, by contrast, is identical to the shear Alfvén wave of the previous section, except with a perpendicular eigenvector $\boldsymbol{\xi}$. For propogation along magnetic field lines, there are three orthogonal modes: one sound wave, and two orthogonal shear Alfvén waves.

$\boldsymbol{\theta = \pi/2}$: When $\theta = \pi/2$, the wave is propagating perpendicular to the magnetic field. In this case, eq. (5.22) reduces to

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + v_A^2}{2} \pm \frac{c_s^2 + v_A^2}{2}$$

which has the two solutions

$$\omega^2 = k^2(v_A^2 + c_s^2), \text{ and } \omega^2 = 0.$$

Since $k_\parallel = 0$, then the shear Alfvén wave has the solution $\omega^2 = 0$ as well. Thus, for propagation perpendicular to the magnetic field, only one mode propagates. This mode, called the fast, magnetosonic, or compressional Alfvén wave, involves a perturbed magnetic field $\boldsymbol{B}_1 \parallel \boldsymbol{B}_0$. The fast wave involves the compression and rarefaction of magnetic field lines, propagating perpendicular to the magnetic field. Unlike the shear Alfvén wave, where magnetic field lines oscillate sinusoidally, for the compressional Alfvén wave magnetic field lines remain straight but vary in magnitude. In the compressional Alfvén wave, magnetic pressure and plasma pressure both contribute to the restoring force allowing the wave to propagate.

Derivation

From linear algebra, we know that the eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Because the matrix $\boldsymbol{M}$ in eq. (5.19) is symmetric, then each of the eigenvectors corresponding to different MHD waves are orthogonal.

Using the coordinate system where $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ and $\boldsymbol{k} = k_\perp\hat{\boldsymbol{x}} + k_\parallel\hat{\boldsymbol{z}}$, we saw that the eigenvector for the shear Alfvén wave was $\boldsymbol{\xi} = \xi_y\hat{\boldsymbol{y}}$. Thus, the eigenvectors for the fast and slow waves must lie in the $x - z$ plane. Letting

$$\boldsymbol{\xi} = \xi_\perp\hat{\boldsymbol{x}} + \xi_\parallel\hat{\boldsymbol{z}},$$

the RHS terms in eq. (5.18) simplify to

$$-\gamma P_0\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\xi}) = -\gamma P_0\boldsymbol{k}(k_\perp\xi_\perp + k_\parallel\xi_\parallel)$$

and

$$-\frac{1}{\mu_0}\Big(\boldsymbol{k}\times\Big(\boldsymbol{k}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\Big)\Big)\times\boldsymbol{B}_0 = \frac{B_0\xi_\perp}{\mu_0}\Big(\boldsymbol{k}\times\Big(\boldsymbol{k}\times\hat{\boldsymbol{y}}\Big)\Big)\times\boldsymbol{B}_0 = -\frac{k^2\xi_\perp B_0}{\mu_0}\hat{\boldsymbol{y}}\times\boldsymbol{B}_0 = -\frac{1}{\mu_0}k^2B_0^2\xi_\perp\hat{\boldsymbol{x}}.$$

The x and z components of eq. (5.18) thus become

$$-\rho\omega^2\xi_\perp = -\gamma P_0 k_\perp(k_\perp\xi_\perp + k_\parallel\xi_\parallel) - \frac{1}{\mu_0}k^2B_0^2\xi_\perp$$

$$-\rho\omega^2\xi_\parallel = -\gamma P_0 k_\parallel(k_\perp\xi_\perp + k_\parallel\xi_\parallel).$$

Using $k_\parallel = k\cos\theta$, $k_\perp = k\sin(\theta)$, $c_s^2 = \frac{\gamma P_0}{\rho}$ and $v_A^2 = \frac{B^2}{\mu_0\rho}$, these can be written as

$$\frac{\omega^2}{k^2}\xi_\perp = (c_s^2\sin^2\theta + v_A^2)\xi_\perp + c_s^2\sin\theta\cos\theta\xi_\parallel \tag{5.23}$$

$$\frac{\omega^2}{k^2}\xi_\parallel = c_s^2(\cos\theta\sin\theta\xi_\perp + \cos^2\theta\xi_\parallel) \tag{5.24}$$

or, in matrix form, as

$$\begin{bmatrix}(c_s^2\sin^2\theta + v_A^2) - \frac{\omega^2}{k^2} & c_s^2\sin\theta\cos\theta \\ c_s^2\sin\theta\cos\theta & c_s^2\cos^2\theta - \frac{\omega^2}{k^2}\end{bmatrix}\begin{bmatrix}\xi_\perp \\ \xi_\parallel\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Setting the determinant to zero gives a quartic equation for ω/k:

$$\frac{\omega^4}{k^4} - (c_s^2 + v_A^2)\frac{\omega^2}{k^2} + v_A^2c_s^2\cos^2\theta = 0$$

with the solutions

$$\frac{\omega^2}{k^2} = \frac{(c_s^2+v_A^2)}{2} \pm \frac{1}{2}\sqrt{(c_s^2+v_A^2)^2 - 4v_A^2c_s^2\cos^2\theta}.$$

Using

$$(c_s^2+v_A^2)^2 - 4v_A^2c_s^2\cos^2\theta = (v_A^2 - c_s^2)^2 + 4v_A^2c_s^2\sin^2\theta$$

this is equivalent to the dispersion relation for fast and slow waves, eq. (5.22).

5.3 Streaming instability

The streaming instability is an instability that arises in cold plasmas where one species is moving with a net velocity relative to other species. The instability is driven by an electrostatic field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$. The dispersion relation for the streaming instability is given by

$$1 = \sum_\sigma \frac{\omega_{p\sigma}^2}{(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{\sigma0})^2}. \tag{5.25}$$

Note that if $\boldsymbol{u}_{\sigma0} = 0$, eq. (5.25) simplifies to a dispersion relation for plasma oscillations with $\omega^2 = \sum_\sigma \omega_{p\sigma^2}$.

As we will show, for a two-component plasma if $\boldsymbol{u}_{\sigma0}$ differs between each species then $\omega^2 < 0$ for sufficiently small k, implying that ω is imaginary and the oscillation grows exponentially. This exponential growth is called an instability. We will not be able to predict the dynamics of this instability, only the criteria that determines the onset of the instability.

Physical intuition

I don't have any physical intuition for what causes the streaming instability.

Derivation

To derive the dispersion relation eq. (5.25), we linearize the multi-fluid equations and Gauss's law around a uniform background state with non-zero velocity for each species, such that

$$n_\sigma(\boldsymbol{r},t) = n_{\sigma0} + \epsilon n_{\sigma1}(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)$$

$$\boldsymbol{u}_\sigma(\boldsymbol{r},t) = \boldsymbol{u}_{\sigma0} + \epsilon\boldsymbol{u}_{\sigma1}(\boldsymbol{r},t) + \mathcal{O}(\epsilon^2)$$

$$\phi(\boldsymbol{r},t) = \phi_1(\boldsymbol{r},t).$$

Linearizing the multi-fluid equations and Gauss's law for electrostatic fields gives

$$\frac{\partial n_{\sigma 1}}{\partial t} + n_{\sigma 0}\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\sigma 1} + \boldsymbol{u}_{\sigma 0}\cdot\boldsymbol{\nabla} n_{\sigma 1} = 0$$

$$m_\sigma \frac{\partial \boldsymbol{u}_{\sigma 1}}{\partial t} + m_\sigma(\boldsymbol{u}_{\sigma 0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\sigma 1} = -q_\sigma \boldsymbol{\nabla}\phi_1$$

$$\frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} = -\boldsymbol{\nabla}^2\phi_1.$$

We then assume that each first-order quantity has exponential dependence $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, which gives

$$-i(\omega + \boldsymbol{u}_{\sigma,0}\cdot\boldsymbol{k})n_{\sigma 1} + i\boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 1} = 0$$

$$-im_\sigma(\omega - \boldsymbol{u}_{\sigma 0}\cdot\boldsymbol{k})\boldsymbol{u}_{\sigma 1} = -iq_\sigma \boldsymbol{k}\phi_1$$

$$\frac{1}{\epsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} = k^2\phi_1.$$

Solving for $\boldsymbol{u}_{\sigma 1}$ and $n_{\sigma 1}$ in terms of ϕ_1 gives

$$\boldsymbol{u}_{\sigma 1} = \frac{\frac{q_\sigma}{m_\sigma}\boldsymbol{k}\phi_1}{\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 0}}$$

$$n_{\sigma 1} = \frac{n_{\sigma 0}\boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 1}}{\omega - \boldsymbol{u}_{\sigma 0}\cdot\boldsymbol{k}} = \frac{k^2 n_{\sigma 0}\frac{q_\sigma}{m_\sigma}\phi_1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 0})^2}.$$

Plugging our expression for $n_{\sigma 1}$ into the linearized Gauss's law gives

$$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_\sigma q_\sigma \frac{k^2 n_{\sigma 0}\frac{q_\sigma}{m_\sigma}}{(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{\sigma 0})^2}\phi_1$$

which simplifies to the dispersion relation eq. (5.25).

5.3.1 Electron-positron streaming instability

To illustrate how eq. (5.25) gives rise to an instability, we consider a plasma made of two species: electrons, with mass m_e and charge $-e$, and positrons, with mass m_e and charge $+e$. A positron-electron plasma could never exist for very long in reality, because the positrons and electrons would quickly annihilate one another. Nevertheless, we will examine this toy model as a way of studying the streaming instability. We will first derive the criteria describing the onset of the instability before calculating the peak growth rate of the instability.

To ensure quasineutrality, the background densities for each species are equal $n_{e0} = n_{p0}$. The streaming instability arises if the species in the plasma have different velocities, so we consider a plasma with background electron velocity

$$\boldsymbol{u}_{e0} = \boldsymbol{u}_0$$

and background positron velocity

$$\boldsymbol{u}_{p0} = -\boldsymbol{u}_0.$$

Equation (5.25) then becomes

$$1 = \frac{1}{(\frac{\omega}{\omega_{pe}} - \frac{\boldsymbol{k}\cdot\boldsymbol{u}_0}{\omega_{pe}})^2} + \frac{1}{(\frac{\omega}{\omega_{pe}} + \frac{\boldsymbol{k}\cdot\boldsymbol{u}_0}{\omega_{pe}})^2}.$$

Defining

$$z = \frac{\omega}{\omega_{pe}}$$

and

$$\lambda = \frac{\boldsymbol{k}\cdot\boldsymbol{u}_0}{\omega_{pe}}$$

the dispersion relation can be written as

$$1 = 1 = \frac{1}{(z-\lambda)^2} + \frac{1}{(z+\lambda)^2}.$$

This is a fourth-order polynomial equation for z which, after some algebra, is

$$z^4 - 2z^2(\lambda^2 + 1) + \lambda^2(\lambda^2 - 2) = 0.$$

Solving for z^2 gives

$$z^2 = (\lambda^2 + 1) \pm \sqrt{(4\lambda^2 + 1)}. \tag{5.26}$$

This gives a negative-z^2 solution if

$$\sqrt{4\lambda^2 + 1} > \lambda^2 + 1$$

which happens for

$$0 < z < \sqrt{2}$$

or

$$0 < ku_0 < \sqrt{2}\omega_{pe}. \tag{5.27}$$

Using the definition of z, negative z^2 corresponds to imaginary ω with both exponentially damped and growing solutions. Thus, exponential growth of the initial perturbations (corresponding to an instability) will occur will arise for k satisfying eq. (5.27).

We now calculate the growth rate of the electron-positron streaming instability. The maximum growth rate occurs at the minimum value of z^2, which happens when

$$\frac{dz}{d\lambda} = 0.$$

Taking the derivative of eq. (5.26) gives

$$2z\frac{dz}{d\lambda} = 2\lambda - \frac{4\lambda}{\sqrt{4\lambda^2 + 1}} = 0.$$

This equation is satisfied at

$$\lambda = \frac{\sqrt{3}}{2}.$$

Thus, the maximum growth rate of the instability occurs at

$$z^2 = \frac{7}{4} - \sqrt{4} = -\frac{1}{4}$$

or at

$$\omega = \pm\frac{\omega_{pe}}{2}.$$

Thus, the growth rate of the instability is nearly as large as the electron plasma frequency. Thus, the streaming instability occurs very quickly in electron-positron plasmas.

5.3.2 Electron-ion streaming instability

We now consider the more realistic situation of electrons moving with velocity $\boldsymbol{u}_0$ past stationary ions. For this situation, the dispersion relation is now

$$1 = \frac{\omega_{pe}^2}{(\omega - \boldsymbol{k} \cdot \boldsymbol{u}_0)^2} + \frac{\omega_{pi}^2}{\omega^2}.$$

Defining the dimensionless variable

$$\epsilon = \frac{m_e}{m_i}$$

and again using the dimensionless variables $z = \omega/\omega_{pe}$ and $\lambda = \boldsymbol{k} \cdot \boldsymbol{u}_0/\omega_{pe}$, this can be written in dimensionless form as

$$1 = \frac{1}{(z - \lambda)^2} + \frac{\epsilon}{z^2}.$$

A somewhat tedious calculation shows that the condition for instability is given by

$$0 < \boldsymbol{k} \cdot \boldsymbol{u}_0 < \omega_{pe}\Big[1 + \Big(\frac{m_e}{m_1}\Big)^{\frac{1}{3}}\Big]^{\frac{3}{2}}. \tag{5.28}$$

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