Classical, Neoclassical, and Anamolous Transport Notes
AST 568

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Contents

0 Introduction

AST568 is divided into two sections. The first half, taught by Professor Hong Qin, introduces the students to differential geometry and gyrokinetics. The second half, taught by Professor Bill Tang, covers plasma transport, including classical, neoclassical, and anomalous transport. These notes are intended to teach and explain the material covered in the second half of AST568 taught at Princeton University during Spring 2018.

I am writing these notes primarily as a learning experience. I learn best when I am forced to explain something to someone else. I am also writing these notes because I felt that the course would benefit greatly from a more thorough set of lecture notes. Hopefully these notes can be a useful reference to future students taking the course. As a student, I tend to learn by working through notes outside of class, and I know that notes like these would have been useful for me as I was taking the class.

These notes are obviously still a work in progress. Once I've corrected these notes and have them in their final version, if you find a typo or an error, no matter how small, please send me an email at mcgreivy@princeton.edu so I can fix it. I'm also highly appreciative of suggestions to make these notes better.

I've divided these notes into four chapters, corresponding to the material covered in class, roughly in the order which it was covered. The first chapter covers classical transport, which is related to the transport of particles and energy in a magnetized cylinder due to collisions. The second chapter covers collisional neoclassical diffusion, while the third chapter covers collisionless neoclassical diffusion. Neoclassical diffusion is the transport of particles and energy in a realistic toroidal geometry, due to the effects of collisions. It's like classical transport, but with a curved magnetic field instead of a straight magnetic field. Neoclassical transport is always higher than classical transport. The final chapter covers anomalous transport, which is enhanced energy and particle transport in a toroidal geometry due to waves and turbulence. This anomalous transport increases the transport over the neoclassical levels alone. Classical transport is the lowest level of transport, followed by neoclassical transport. Anomalous transport enhances the transport levels above the neoclassical level. Tokamaks unfortunately have anomalous as well as neoclassical transport.

1 Classical Transport

"We've already figured out fusion. We just need an infinitely long cylinder."

Bob Kaita

Heat flows from hot to cold. If we want to do fusion on earth, we're going to have to make some gas really hot. If we want to create commercial amounts of fusion energy from magnetically confined plasma, we need to *keep* that plasma hot, mere meters away from the room temperature outside world. These notes concern themselves with the question of how well a magnetically confined plasma keeps itself hot. This subject is called 'plasma transport', or sometimes just 'transport'. Transport is an enormously important topic in fusion because if we don't achieve sufficiently low levels of transport, magnetic fusion will be, at best, uneconomical. For the purposes of AST568, there are three types of transport we concern ourselves with: classical, neoclassical, and anomalous. This chapter focuses on classical transport, which is the transport of a plasma in a cylindrical geometry due to collisions between particles. Classical diffusion is really slow - if magnetic confinement devices transported heat and particles out at the classical rate, we would've reached ignition a long time ago. Unfortunately, we haven't. There are a number of reasons for this, and over the course of these notes we'll begin to understand these reasons.

In these notes, we are going to focus on tokamaks, as well as hydrogen ($q_i = +e$) plasmas. Neoclassical and anamolous transport in stellerators are important topics in themselves, but not ones that are covered in class.

1.1 Diffusion

Let's say a few words about diffusion before we get to the plasma physics. When microscopic particles move randomly in small steps, the particle flux $\vec{\Gamma}$ is usually proportional to the density gradient.

$$\vec{\Gamma} = -D\vec{\nabla}n \tag{1.1}$$

The constant of proportionality D is the called diffusion coefficient. Since the change in time of density at a point in space with is proportional to the divergence of the particle flux at that point, we have

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \vec{\Gamma} \tag{1.2}$$

This is just a continuity equation for particles. Putting together equations 1.1 and 1.2, we have the diffusion equation for the evolution of particle density.[1]

$$\frac{\partial n}{\partial t} = D\vec{\nabla}^2 n \tag{1.3}$$

[1] Although I am focusing on particle density here, this equation is also known as the *heat equation*. When thermal energy diffuses in a random-walk process, the temperature obeys a diffusion equation as well.

This is a pretty intuitive equation. It tells us that if the density of particles is constant in space, it won't change. It also tells us that if the density of particles changes linearly with position, the density at each point in space won't change even though we have a non-zero flux. For the density at each point in space to change with time, there needs to be some non-linear change of density with position. What about D, the diffusion coefficient? How do we calculate D? Well, one way to do it is to suppose that our particles move in a random-walk, with a step size Δx in some random direction every time interval Δt. If we were to calculate the diffusion coefficient for a random walk, we find the diffusion coefficient goes as Todo: look at 401 notes to see how this is calculated

$$D \sim \frac{(\Delta x)^2}{\Delta t} \tag{1.4}$$

We'll be using this result a bunch throughout these notes. We also have that the root-mean squared displacement of a random-walk process, $\sqrt{\langle x^2 \rangle}$, goes like Todo: look at 401 notes to see how this is calculated

$$\sqrt{\langle x^2 \rangle} \sim \sqrt{2NDt} \tag{1.5}$$

where N is the number of dimensions the particle can diffuse in. We'll often write the LHS as simply $\sqrt{Dt}$. What this equation tells us is that if we look at a single particle undergoing a random-walk, then after time t we can roughly expect that the particle is a distance $\sim \sqrt{Dt}$ away from where it started. Of course, the average distance away is zero (since half move in the positive direction and half move in the negative direction), so we look at the rms displacement. Classical transport attempts to calculate the perpendicular diffusion coefficient for a magnetically confined plasma. If a plasma is magnetically confined, the particles orbit around the magnetic field. When they collide with other particles, their guiding center position changes. Assuming the collisions happen randomly, then the diffusion coefficient will be, approximately, the change in the gyrocenter position between collisions squared divided by the average time between collisions. Before we calculate the diffusion coefficent for classic transport, let's discuss why in classical transport, we are generally more interested in electron-ion collisions than ion-ion or electron-electron collisions.

1.2 Like and Unlike Particle Collisions

It turns out, as we will show in a moment, that collisions between electrons and ions in a magnetized plasma will lead to the center of their orbits (gyro-centers) being displaced the same amount, in the same direction. On the other hand, for *like*-particle collisions (electron-electron and ion-ion collisions), the gyro-centers $\vec{r}_{gc}$ of the two particles get displaced *oppositely* from each other. We can write this mathematically, for a two-particle collision, as

$$q_1 \Delta \vec{r}_{gc1} + q_2 \Delta \vec{r}_{gc2} = 0 \tag{1.6}$$

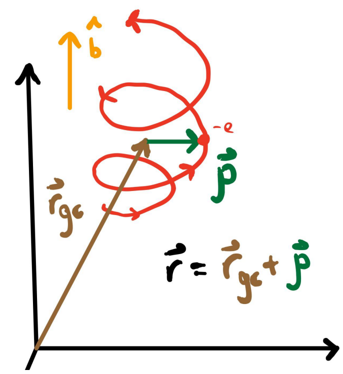

Figure 1: The division of a particle's position into the guiding center $\vec{r}_{gc}$ and the gyro-radius $\vec{\rho}$.

This curious observation has a few important consequences. Firstly, we should be mainly concerned with electron-ion collisions when studying diffusion in magnetized plasmas, and not like-particle collisions. The reason for this is that like-particle collisions do not have a *net* diffusion. While the density of particles *can* evolve due to like-particle collisions,[2] each particles motion is not truly random because it is necessarily opposite to the motion of another particle, and thus the evolution of density doesn't follow equation 1.1. If the density evolution doesn't follow equation 1.1, then we can't use the diffusion equation for like-particle collisions. Also, the evolution of the density will be much smaller when the net diffusion is zero. Therefore, we will concentrate on ion-electron collisions for calculating diffusion coefficients. The second consequence of our curious observation is that the rate of diffusion of electrons and ions due to displacement of their gyro-centers is *equal*,[3] since their step size and time between collisions is the same for each species. We therefore don't have to worry about separate diffusion coefficients for electrons and ions. This discussion is only true for classical diffusion, and is modified for neoclassical diffusion. But we'll have to wait to chapter 2 to see how this gets modified.

Let's derive this result. Suppose we have a constant magnetic field pointing in the z-direction, and a particle of mass m, charge q, and velocity $\vec{v}$ orbiting in the field. The equation of motion for the particle is just $m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$. We know the solution to this equation - the solution is gyromotion perpendicular to the field and free motion along the field. We can write the position $\vec{r}$ as the

[2] For example, imagine a delta-function density function of ions in a magnetized plasma. Even if the ions only collide with other ions during their gyro-orbits, $n(x)$ can still change due to like-particle collisions. It just isn't a classic random-walk process, so it doesn't follow the diffusion equation. It's not a classic random-walk process, because the flux of particles isn't proportional to the a density gradient. If there is a density gradient, for ion-ion collisions there will be as many particles diffusing up the gradient as down the gradient.

[3] At least, for hydrogen plasmas where $q_i = e$, which is what we're considering in these notes.

sum of the guiding center position $\vec{r}_{gc}$ and the gyro-radius vector $\vec{\rho}$, as we did in GPP1 and as shown in figure 1.

$$\vec{r} = \vec{r}_{gc} + \vec{\rho} = \vec{r}_{gc} + \frac{m\hat{b} \times \vec{v}}{qB}$$

The gyro-center $\vec{r}_{gc}$ is

$$\vec{r}_{gc} = \vec{r} + \frac{m\vec{v} \times \hat{b}}{qB} \tag{1.7}$$

So far, this is all stuff from GPP1. Now, in a collision, momentum is conserved. The change in momentum of the first particle in the collision plus the change in momentum of the second particle in the collision is zero.

$$m_1 \Delta\vec{v}_1 + m_2 \Delta\vec{v}_2 = 0$$

Taking the cross product of this equation with $\vec{b}$, we get

$$m_1 \Delta\vec{v}_1 \times \hat{b} + m_2 \Delta\vec{v}_2 \times \hat{b} = 0 \tag{1.8}$$

If the collision is fast, we also have that the position of each particle doesn't change during a collision, or $\Delta\vec{r} = 0$. Thus, from equation 1.7 the change in gyro-center during a collision is

$$\Delta\vec{r}_{gc} = \frac{m\Delta\vec{v} \times \hat{b}}{qB} \tag{1.9}$$

Combining this with the conservation of momentum, equation 1.8, we get

$$q_1 \Delta\vec{r}_{gc1} + q_2 \Delta\vec{r}_{gc2} = 0 \tag{1.10}$$

This is the promised result - particles of the same charge diffuse in opposite directions (i.e. no net diffusion) which particles of different charge (i.e. electrons and ions) diffuse in the same direction, with the the same magnitude, at least for hydrogen. Remember what this conclusion tells us. Firstly, it tells us that for the purposes of calculating diffusive transport, we should concern ourselves ourselves with electron-ion collisions, and not ion-ion or electron-electron collisions. Secondly, it tells us that electrons and ions diffuse at the same rate.

As long as the ions are singly-ionized, this result (equation 1.10) doesn't depend on the mass of the particles involved. However, if the ions are not singly-ionized but multiply-ionized, then equation 1.10 tells us that the electrons would diffuse at a faster rate than the ions. Question: how does it work itself out in this case? Is it some sort of modified ambipolar diffusion?

1.3 Heuristic Estimate of Classical Diffusion Coefficient

As I've mentioned, classical transport concerns itself with diffusion of particles in a straight, magnetized cylinder due to collisions. Before we calculate the diffusion coefficient more rigorously, let's try to see if we can get a simple heuristic estimate of the classical diffusion coefficient. Look back at equation 1.4. If we know the average change in the gyro-center after each collision (Δx) and the average time between collisions (Δt), we have an estimate of the diffusion coefficient. Well, in a straight cylinder, the timescale between collisions Δt is approximately the electron-ion collision time $1/\nu_{ei}$ where[4]

$$\nu_{ei} = \frac{4\pi n e^4}{m_e^2 V_{Te}^3} \ln \Lambda \tag{1.11}$$

and $\Lambda = \frac{4\pi}{3} n \lambda_D^3$ is the number of particles in a Debye sphere. The change in the guiding-center position between collisions Δx is approximately the electron gyro-radius, ρ_e, where

$$\rho_e = \frac{m_e V_{Te}}{eB} \tag{1.12}$$

How do we know it's the electron gyro-radius as the step size instead of the ion gyro-radius or some other distance? This comes down to the fact that the ions are so much heavier. Since ions are so heavy compared to electrons, their velocity only changes by a tiny amount compared to an electron in a collision. But the electron's velocity, in an elastic collision with the effectively infinitely massive ion, changes it's velocity by an amount of order the electron velocity. This can we written as $\Delta v_e \sim v_e$. Thus, from equation 1.9, we can see that the change in the electron gyro-center position between collisions is about the electron gyroradius. This means also that the ion gyro-center changes by ρ_e as well, since their gyro-centers change by the same amount in a collision as we saw in section 1.2. Having an estimate for Δx and Δt, we estimate our classical diffusion coefficient to be

$$D_{\text{class}} = \frac{(\Delta x)^2}{\Delta t} = \rho_e^2 \nu_{ei} \tag{1.13}$$

The key concept is that the step size is the electron gyro-radius, and the timescale is the ion-electron collision time.

1.4 Classical Diffusion Coefficient: Multi-Fluid

Classic transport relates to the diffusion of particles and energy in a straight, time-independent, collisional, magnetized cylinder. The picture to have in your mind is a plasma in the geometry of an infinitely long, θ-symmetric and z-symmetric cylinder, with no perturbations or waves in the plasma. There is some r-dependent density profile, and the only way this density profile changes is from collisions. This is sketched in figure 2. We'll derive the classical diffusion coefficient two separate ways - first, using the multi-fluid model and second, using resistive (i.e. highly collisional) MHD[5]. In this subsection, we'll derive the classical diffusion coefficient using the multi-fluid model.

[4] Question: units of this? Get in non-gaussian units

[5] If we used ideal MHD, we would zero resistivity, and thus we would have the frozen-flux theorem so there would be no diffusion. For that reason we use resistive MHD.

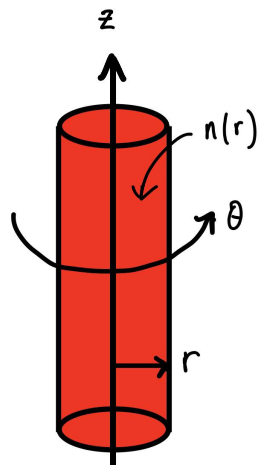

Figure 2: An infinitely long cylindrically symmetric straight cylinder of plasma in a straight magnetic field pointing in the z-direction. There is a radially-dependent density profile $n(r)$. This is the geometry used to study classical diffusion.

To get a diffusion coefficient, we want to somehow relate the particle flux $\vec{\Gamma}$ to the density gradient, as in equation 1.1. More specifically, in a cylindrical geometry, we want an equation for the radial particle flux Γ_r, as this will tell us the transport coefficient *across* the magnetic field, which is what we are interested in.[6] This will be our goal - to get an equation for the radial particle flux, $\Gamma_r \equiv nv_r$, in terms of the density gradient. We'll start with the multi-fluid equation of motion. The general multi-fluid equation of motion is, as we know from GPP1,

$$m_\sigma n_\sigma \frac{\partial \vec{u}_\sigma}{\partial t} + m_\sigma n_\sigma (\vec{u}_\sigma \cdot \vec{\nabla})\vec{u}_\sigma = q_\sigma n_\sigma \vec{E} + q_\sigma n_\sigma \vec{u}_\sigma \times \vec{B} - \vec{\nabla} \cdot \overleftrightarrow{P}_\sigma + \sum_{\alpha \neq \sigma} \vec{R}_{\sigma\alpha} \quad (1.14)$$

We can make a few simplifications immediately for our cylindrical geometry. Since our cylindrical geometry is assumed to have no time-dependence, we can get rid of the partial-derivatives with respect to time. We'll also assume any velocities are small, so we can ignore the convective derivative term. We'll assume there are no macroscopic electric fields in our cylinder. We can argue that this E-field assumption makes sense based on (i) net-neutrality of a plasma over large scales, and (ii) the fact that electrons and ions diffuse at the same rate in classical diffusion. We also assume that our plasma is collisional. In this case, we can replace the pressure tensor with just a single scalar pressure, so

[6] I suppose we could try to calculate the poloidal particle flux, Γ_θ, as this would also involve transport perpendicular to the magnetic field. This would be a lot harder to calculate though, because we'd have to then assume that our density is not cylindrically symmetric.

that

$$\vec{\nabla} P_\sigma = \vec{\nabla} P = \frac{dP}{dr}\hat{r}$$

It is also true that

$$\vec{R}_{ei} = -\nu_{ei} m_e n_e (\vec{u}_e - \vec{u}_i) = -\vec{R}_{ie}$$

where ν_{ei} is the electron-ion collision frequency, or the inverse of the electron-ion collision time. Note that I've defined $\vec{R}_{\sigma\alpha}$ to be the force per volume on species σ due to collisions with species α. In irreversibles, we'll calculate this to be

$$\nu_{ei} = \frac{4\pi n e^4}{m_e^2 v_{Te}^3} \ln \Lambda \tag{1.15}$$

This is a result we"ll examine in more detail in section 1.4.1. Lastly, we'll assume that the density and pressure of each species are equal, although the velocities are not necessarily equal. With all of these assumptions, we're left with three terms in our multi-fluid momentum equations for the electrons and the ions.

$$\begin{aligned} \frac{dP}{dr}\hat{r} &= -en\vec{u}_e \times \vec{B} - \nu_{ei} m_e n (\vec{u}_e - \vec{u}_i) \\ \frac{dP}{dr}\hat{r} &= en\vec{u}_i \times \vec{B} - \nu_{ei} m_e n (\vec{u}_i - \vec{u}_e) \end{aligned} \tag{1.16}$$

Before we go any further, let's stop for a moment and ask ourselves what these equations are telling us physically. We have a radial pressure gradient balanced by two terms, a magnetic force and an ion-electron collision force. We know that $u_{ri} = u_{re}$ because the diffusion rates of ions and electrons must be the same. Thus, the radial component of the friction force can't balance the radial pressure gradient - the magnetic force has to. Since $\vec{B}$ is in the z-direction, we need there to be some θ-component of the velocities for the magnetic force to balance the pressure gradient. Since the electrons and ions have different charges, u_θ needs to be in opposite directions for the ions and electrons. Thus, the physical situation compatible with this equation is of each species having a θ-velocity opposite in direction to the other species, acting in some relatively complicated way against the pressure gradient and the collision force. With this physical understanding, let's solve these equations. Writing the equations in component form, we have

$$\frac{dP}{dr} = -enu_{\theta e} B \tag{1.17}$$

$$\frac{dP}{dr} = enu_{\theta i} B \tag{1.18}$$

$$0 = -enu_r B - \nu_{ei} m_e n (u_{\theta i} - u_{\theta e}) \tag{1.19}$$

Notice that I've set $u_{ri} = u_{ei} = u_r$, since I know the ions and electrons diffuse at the same rate. The first two equations tell us that $u_{\theta i} = -u_{\theta e}$. Thus, from equation 1.19,

$$enu_r B = -2\nu_{ei} m_e n u_{\theta i}$$

$$u_{\theta i} = -\frac{eBu_r}{2\nu_{ei}m_e} \tag{1.20}$$

Plugging this into equation 1.18, we have

$$\frac{dP}{dr} = -\frac{e^2B^2}{2m_e\nu_{ei}}nv_r \tag{1.21}$$

Using the ideal gas law $P = 2nk_BT$, which is valid in a collisional regime, and assuming the temperature gradient is zero for simplicity, we have

$$nv_r = -\frac{4m_e\nu_{ei}k_BT}{e^2B^2}\vec{\nabla}n$$

Using $V_{Te}^2 = \frac{2k_BT_e}{m_e}$ and $\rho_e^2 = \frac{m_e^2V_{T,e}^2}{e^2B^2}$, this can be rewritten as

$$\Gamma_r = -2\rho_e^2\nu_{ei}\vec{\nabla}n = -2D_{\text{class}}\vec{\nabla}n \tag{1.22}$$

where $D_{\text{class}} = \rho_e^2\nu_{ei}$. This is the classical diffusion coefficient.

Question: How does bill do it differently in the notes in class? It isn't this way..

1.4.1 The Coulomb Collision Operator and Collisional Momentum Transfer

In the multi-fluid equations, we have a term $\vec{R}_{\sigma\alpha}$ which represents the momentum per second per volume (force per volume) transferred to species σ due to collisions with species α. This term is defined as

$$\vec{R}_{\sigma\alpha} = m_\sigma\int C(f_\sigma, f_\alpha)\vec{v}d^3\vec{v} \tag{1.23}$$

where $C(f_\sigma, f_\alpha)$ is the collision operator for collisions between species σ and species α. From this definition, make sure you understand why this term represents the momentum per volume transferred to species σ due to collisions. In GPP1, we argued that it would make sense for this term to be proportional to the velocity difference between species σ and α, and asserted that we could show that

$$\vec{R}_{\sigma\alpha} = -m_\sigma n_\sigma\nu_{\sigma\alpha}(\vec{u}_\sigma - \vec{u}_\alpha) \tag{1.24}$$

We've used this result earlier in this subsection. However, we've never actually derived this result. It turns out it can be derived from the Coulomb collision operator, using a Maxwellian velocity distribution function for the ions and electrons whose average velocity is shifted to $\vec{u}_i$ or $\vec{u}_e$. We derive this result in the first homework. For now, let's derive the Coulomb collision operator, starting from the Vlasov-Maxwell equation and the Fokker-Planck operator.[7]

[7] We haven't yet derived the Fokker-Planck operator, so I'm not sure what the assumptions of the Fokker-Planck operator are. I think the collision model used by this operator is fairly general. This operator gets derived in irreversibles. What Bill wants us to know is that the first term represents a 'drag force', while the second term represents a the effect of diffusion in velocity space. Question: what else should we know?

The Fokker-Planck operator is a pretty general collision operator which takes into account the effect of ion-electron collisions on the electron distribution function. The Fokker-Planck operator is

$$C_{fp}(f_e, f_i) = \nu_{ei} V_{Te}^3 \Big[-\frac{\partial}{\partial v_\alpha}\big(f_e \frac{\partial h_i}{\partial v_\alpha}\big) + \frac{1}{2}\frac{\partial}{\partial v_\alpha}\frac{\partial}{\partial v_\beta}\big(f_e \frac{\partial^2 g_i}{\partial v_\alpha \partial v_\beta}\big)\Big] \tag{1.25}$$

where Einstein summation notation has been used. We have that

$$V_{Te} = \Big(\frac{2k_B T_e}{m_e}\Big)^{\frac{1}{2}}$$

and

$$h_i(\vec{x}, \vec{v}, t) = \frac{m_e + m_i}{m_i}\frac{1}{n_i(\vec{x})}\int \frac{f_i(\vec{x}, \vec{v}', t)}{|\vec{v} - \vec{v}'|} d^3\vec{v}' \tag{1.26}$$

$$g_i(\vec{x}, \vec{v}, t) = \frac{1}{n_i(\vec{x})}\int |\vec{v} - \vec{v}'| f_i(\vec{x}, \vec{v}', t) d^3\vec{v}' \tag{1.27}$$

$$\nu_{ei} = \frac{4\pi n e^4}{m_e^2 v_{Te}^3}\ln\Lambda \tag{1.28}$$

The terms h_i and g_i are named the "Rosenbluth Potentials".[8] Note that the factors of $\frac{1}{n_i(\vec{x})}$ are necessary for the Fokker-Planck operator to have the correct units.[9] For general f_i, this is either tricky or impossible to solve. However, if $m_i \gg m_e$, then since the electrons have a thermal velocity $\sqrt{\frac{m_i}{m_e}} \sim 40$ times larger than the ions, then the ions are effectively stationary on the timescales of the electrons. More precisely, while the electron and ion distribution functions have some spread in velocity space, the spread of the electrons is inevitably much larger than the spread of the ions in velocity space, so relative to the electron distribution function the ion distribution function looks like a delta-function distribution in velocity space. Thus, we can approximate the ions as having a delta-function potential, with a net velocity $\vec{u}_i$.

$$f_i(\vec{x}, \vec{v}, t) = n_i(\vec{x})\delta^{(3)}(\vec{v} - \vec{u}_i(\vec{x}))$$

With this approximation, the Rosenbluth potentials reduce to

$$h_i(\vec{x}, \vec{v}, t) \approx \frac{1}{|\vec{v} - \vec{u}_i(\vec{x})|} \tag{1.29}$$

$$g_i(\vec{x}, \vec{v}, t) \approx |\vec{v} - \vec{u}_i(\vec{x})| \tag{1.30}$$

We can make an additional simplification to the Fokker-Planck equation if we define the variable $\vec{w} \equiv \vec{v} - \vec{u}_i$. Making this simplification, the derivatives with

[8] Named after the great plasma physicist Marshall Rosenbluth, who presumably derived this collision operator for a plasma.

[9] Question: this is indeed the case, no?

respect to the components of $\vec{v}$ can be replaced with derivatives with respect to the components of $\vec{w}$. The Fokker-Planck operator simplifies to

$$C(f_e, f_i) = \nu_{ei} V_{Te}^3 \Big[-\frac{\partial}{\partial w_\alpha} \big(f_e \frac{\partial}{\partial w_\alpha} \frac{1}{|w|} \big) + \frac{1}{2} \frac{\partial}{\partial w_\alpha} \frac{\partial}{\partial w_\beta} \big(f_e \frac{\partial^2 |w|}{\partial w_\alpha \partial w_\beta} \big) \Big] \tag{1.31}$$

We also have that

$$\frac{\partial}{\partial w_\alpha} \frac{1}{|w|} = \frac{\partial}{\partial w_\alpha} \frac{1}{\sqrt{w_x^2 + w_y^2 + w_z^2}} = -\frac{w_\alpha}{(w_x^2 + w_y^2 + w_z^2)^{\frac{3}{2}}} = -\frac{w_\alpha}{|w|^3}$$

$$\frac{\partial^2}{\partial w_\alpha \partial w_\beta} |w| = \frac{\partial}{\partial w_\alpha} \frac{w_\beta}{|w|} = \frac{\delta_{\alpha\beta}}{|w|} - \frac{w_\alpha w_\beta}{|w|^3}$$

which allows our collision operator to simplify further to

$$C(f_e, f_i) = \nu_{ei} V_{Te}^3 \frac{\partial}{\partial w_\alpha} \Big[\big(f_e \frac{w_\alpha}{|w|^3} \big) + \frac{1}{2} \frac{\partial}{\partial w_\beta} \Big(f_e \big(\frac{\delta_{\alpha\beta}}{|w|} - \frac{w_\alpha w_\beta}{|w|^3} \big) \Big) \Big]$$

We only need a few more simplifications to turn this into the Coulomb operator. With some liberal use of the chain rule and einstein notation, we get

$$\frac{\partial}{\partial w_\beta} \big(f_e \frac{\delta_{\alpha\beta}}{|w|} \big) = \frac{\delta_{\alpha\beta}}{|w|} \frac{\partial f_e}{\partial w_\beta} - f_e \frac{w_\alpha}{|w|^3}$$

$$\begin{aligned} \frac{\partial}{\partial w_\beta} \Big(f_e \frac{w_\alpha w_\beta}{|w|^3} \Big) &= \frac{w_\alpha w_\beta}{|w|^3} \frac{\partial f_e}{\partial w_\beta} + f_e \frac{\delta_{\alpha\beta} w_\beta}{|w|^3} + 3 f_e \frac{w_\alpha}{|w|^3} - 3 f_e \frac{w_\alpha w_\beta w_\beta}{|w|^5} \\ &= \frac{w_\alpha w_\beta}{|w|^3} \frac{\partial f_e}{\partial w_\beta} + f_e \frac{w_\alpha}{|w|^3} \end{aligned}$$

We can now see that the three $f_e \frac{w_\alpha}{|w|^3}$ terms in the simplified collision operator cancel each other, and we are left with our Coulomb collision operator.

$$C_{coul}(f_e, f_i) = -\frac{\nu_{ei}}{2} V_{Te}^3 \frac{\partial}{\partial w_\alpha} \Big[\Big(\frac{\delta_{\alpha\beta}}{|w|} - \frac{w_\alpha w_\beta}{|w|^3} \Big) \frac{\partial f_e}{\partial w_\beta} \Big] \tag{1.32}$$

This is the result we wanted - the Coulomb collision operator. Remember that $\vec{w} = \vec{v} - \vec{u}_i$. This is the collision operator acting on electrons as they collide with ions. Note that this has the correct units, $(L^3 V^3 T)^{-1}$. In the first homework assignment, we will show that if our electron distribution function is Maxwellian with mean velocity $\vec{u}_e$, this equation gives us our equation for the collisional force between particles, equation 1.24.

1.5 Classical Diffusion Coefficient: MHD

We've calculated the classical diffusion coefficient using a heuristic estimate, as well as using the multi-fluid model. We can also calculate the classical diffusion coefficient using MHD. Let's do that now. Let's assume, as in figure 2, that we have a time-independent, θ-symmetric, and z-symmetric plasma in a magnetized infinite cylinder. Because of the symmetry of the problem, we only need to use the following two MHD equations:

$$\vec{J} \times \vec{B} = \vec{\nabla} P \tag{1.33}$$

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{J} \tag{1.34}$$

where $\eta = \frac{m_e}{ne^2}\nu_{ei}$. Let's write this equation in components.

$$\frac{dP}{dr} = J_\theta B \tag{1.35}$$

$$J_r B = 0 \tag{1.36}$$

$$E_\theta - u_r B = \eta J_\theta \tag{1.37}$$

$$E_r + u_\theta B = \eta J_r \tag{1.38}$$

Since we have θ-symmetry and time-independence, E_θ must be zero. This comes from Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$.[10] Thus,

$$J_\theta = -\frac{B u_r}{\eta}$$

Plugging this into equation 1.33, using the ideal gas law such that $P = 2nk_BT$, and assuming constant temperature, we have

$$2k_B T \vec{\nabla} n = -\frac{B^2 u_r}{\eta}$$

Using $\eta = \frac{m_e}{e^2 n}\nu_{ei}$, this becomes

$$n u_r = -\frac{2 m_e k_B T}{e^2 B^2}\nu_{ei}\vec{\nabla} n$$

$$\Gamma_r = -2\rho_e^2 \nu_{ei} \vec{\nabla} n = -2 D_{\text{class}} \vec{\nabla} n \tag{1.39}$$

We've now calculated the classical diffusion coefficient using three methods - a heuristic model, the multi-fluid model, and now resistive MHD. The classical diffusion coefficient is very low. If we could achieve classical transport levels in a magnetic confinement device, we would have reached ignition a long time ago. Unfortunately, the toroidal geometry in a tokamak changes the particle orbits, so that the transport levels are much higher than classical. Even if a tokamak had zero turbulence, the lowest possible transport levels are neoclassical rather than classical.

[10] We could also justify getting rid of E_θ by taking the flux surface (constant-r) average over the cylinder, and not assuming θ-symmetry to calculate the *average* diffusion coefficient. That is how Professor Tang does it in his lecture notes. It's six or a half-dozen, the result is the same.

Todo: discuss differences in physical model of multi-fluid classical transport and MHD classical transport. Are they consistent? If not, why not? Why a factor of 2 difference?

Note that the classical diffusion coefficient goes as $1/B^2$. This would be great for confinement, because we could make significant improvements in the confinement by increasing the magnetic field. In real life, increasing the magnetic field *does* have a big effect on improving confinement and the feasibility of a fusion reactor. Unfortunately, it doesn't have quite as strong an effect as the $1/B^2$ dependence we calculate here.

2 Collisional Neoclassical Transport

"With a faucet running at 15 gallons of water a minute, you have the fuel to supply all the energy in the United States that's used today. Therefore, it's up to the physicist to figure out how to liberate us from the need for having energy. And it can be done, in practice."

RICHARD FEYNMAN

In the previous chapter, we studied the transport of charged particles due to collisions in an infinitely long cylindrical geometry. Infinitely long anything is a bit unrealistic given budget limitations. Fusion requires understanding how plasmas work in real life. Let's consider the more realistic situation of a plasma confined by a toroidally shaped magnetic field. The transport of charged particles due to collisions in a toroidal geometry with a realistic magnetic field configuration is called neoclassical transport. These notes will only consider neoclassical transport in *tokamaks*, rather than stellerators. There are two regimes of interest for neoclassical transport theory. The first is the highly collisional regime, where a fluid description of the plasma can be used. The second description is the low-collisionality regime, where we study the motion of individual particles with long mean free paths, using a kinetic model. In this chapter, we study the collisional plasma regime. In the next chapter, we study the collisionless plasma regime.

We'll start this chapter by introducing the toroidal coordinate system. We'll then estimate the neoclassical diffusion coefficient using basic physical arguments, to preview where we're going and also to give us a bit of physical intuition for the basics of neoclassical diffusion. Lastly, we'll calculate the neoclassical diffusion coefficient for a collisional tokamak using resistive MHD. Todo: fix this paragraph once I figure out what is going on with like and unlike particle collisions

In a real tokamak, particles are not highly collisional. Their mean free path is $\sim$ 1km, while the system size in a tokamak is of course much smaller. This means that there is no such thing as the highly collisional neoclassical transport regime. However, we want to understand the collisional regime before we go onto the more difficult to solve collisionless regime. The diffusion coefficient for collisional neoclassical diffusion is

$$D_{Neo} = 2\nu_{ei}\rho_e^2(1+q^2) = 2D_{Class}(1+q^2) \tag{2.1}$$

where q is the safety factor. The safety factor is the number of times a field line goes around toroidally before it goes around once poloidally. Typically, $q \gg 1$, which means that $D_{Neo} \gg D_{Class}$.

2.1 Like and Unlike Particle Collisions

Question: What is deal with collisions? Why are we interested in electrons?

2.2 Toroidal Coordinate Systems

If we're going to understand diffusion in a Tokamak, we're going to need to find a coordinate system to work with. We could use a cylindrical coordinate system, which works fine for many purposes. However, it isn't going to work fine for our purposes. In this class, we'll use a brand-new coordinate system we've never seen before. I call this coordinate system a toroidal coordinate system, which is a rather logical name for it considering it describes coordinates in a torus. In a toroidal coordinate system, our three coordinates are r, ζ, and θ,[11] such that $\vec{x} = (r, \zeta, \theta)$. This coordinate system is illustrated in figure 3. In order to use this coordinate system, we need to first choose a major radius R_0. With a major radius defined, ζ becomes the toroidal angle of rotation around the z-axis. This is the same as the angle θ in cylindrical coordinates. At each ζ in our toroidal coordinate system, we take a poloidal slice centered around the major radius R_0 and use a cylindrical coordinate system for this poloidal slice. The center of the cylindrical coordinate system is the major radius R_0. The coordinate r is the distance from the major radius to $\vec{x}$, while the angle θ is the angle between $\vec{x}$ and the outwards direction, where the outwards direction is $\hat{\zeta} \times \hat{z}$. Make sure you understand the toroidal coordinate system - the rest of these notes aren't going to make very much sense if you don't know what coordinate system we are working in.

A few comments about the toroidal coordinate system. Firstly, each point in space is no longer uniquely defined. If we wanted to uniquely define each point in space, we could technically restrict ζ from 0 to π. However, we're not going to do that, because that would just be dumb. For example, if we have a point which lies on the axis of the major radius at $\zeta = \frac{3\pi}{2}$, we want to be able to label the point $(r, \zeta, \theta) = (0, 3\pi/2, 0)$ rather than have to label it $(r, \zeta, \theta) = (2R_0, \pi/2, -\pi)$. It might make more sense to restrict $r\cos\theta > -R_0$ or $r < a$ where a is the minor radius of the tokamak, so that each point is uniquely defined in a way that makes sense. However, whether that is useful or not depends on what application we have in mind with our coordinate system, so I won't insist that we do that. Secondly, we have to define a direction for positive θ. You can see in figure 3 how I've defined it. This means that positive-θ for our poloidal cross-section points just like it normally would in a cylindrical coordinate system. Thirdly, we need to redefine our definitions of distance and of the $\vec{\nabla}$ operator in this funky coordinate system. The infinitesimal distance between two points in a toroidal coordinate system is

$$dx^2 = dr^2 + r^2 d\theta^2 + R^2 d\zeta^2$$

where $R = R_0 h$, $h = 1 + \epsilon\cos\theta$ and $\epsilon = \frac{r}{R_0}$. Intuitively, R is the distance from the vertical z-axis to $\vec{x}$, as shown in figure 3. h is just the ratio R/R_0. If we did some fancy math, we could show that the divergence of a vector $\vec{A}$ in toroidal

[11] Technically, the name of the greek letter ζ is zeta. I can never remember the names of the rarely-used greek letters ζ and ξ (xi), so I just call them squiggly. Unfortunately the rest of the scientific community doesn't seem to have adopted this practice. I'm hoping it catches on.

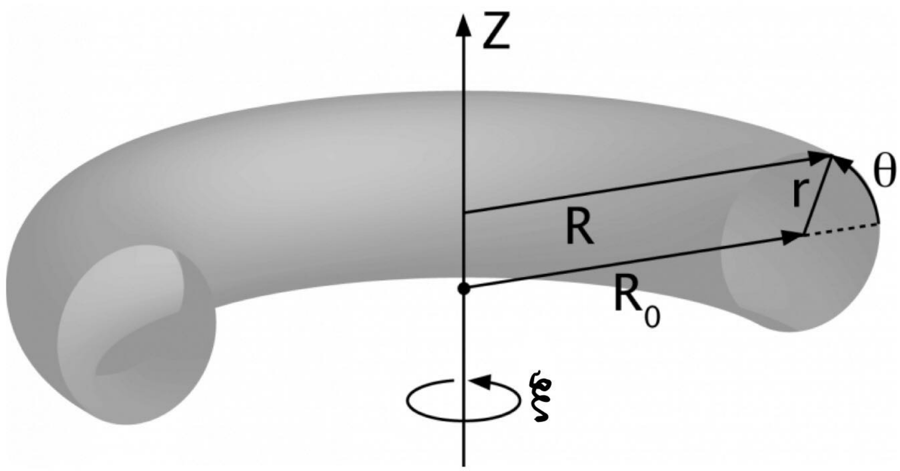

Figure 3: Illustration of the toroidal coordinate system used throughout these notes. In a toroidal coordinate system, there is a fixed major radius R_0. The three coordinates are ζ, the angle around the z-axis, r, the distance to the major radius, and θ, the angle with respect to the major radius. Our coordinate system is right-handed, in that $\hat{\theta} \times \hat{r} = \hat{\zeta}$, $\hat{r} \times \hat{\zeta} = \hat{\theta}$, and $\hat{\zeta} \times \hat{\theta} = \hat{r}$.

coordinates is

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{rh}\left[\frac{\partial}{\partial r}(rhA_r) + \frac{\partial}{\partial \theta}(hA_\theta) + \frac{r}{R_0}\frac{\partial}{\partial \zeta}A_\zeta\right] \tag{2.2}$$

Question: what is gradient in toroidal coordinates? What is curl? These end up being important.

We will want to calculate averages of quantities over flux surfaces in our tokamak. A flux surface for our purposes is the surface formed by taking all the points on the torus at constant r. Fortunately, when we're worried about neoclassical transport theory in Tokamaks, tori are perfectly symmetric in ζ. Thus, to get the average of some quantity A over a flux surface, we only need to average A over θ at constant r. However, this isn't as simple as just integrating $Ad\theta$ and dividing by 2π. This is because there is more surface area outside the tokamak than inside the tokamak. Algebraically, this means we need to multiply by a factor of h when doing the integration. Geometrically, our infinitesimal area elements look like long connected ribbons which go around the torus at constant θ with length $2\pi R = 2\pi R_0 h$ and width $rd\theta$. This is a bit tricky to visualize, so in figure 4 I've tried to illustrate this.[12] Thus, our infinitesimal area element dS is length times width, or $dS = 2\pi Rrd\theta$. So if we want to take

[12] This is also a bit tricky to illustrate.

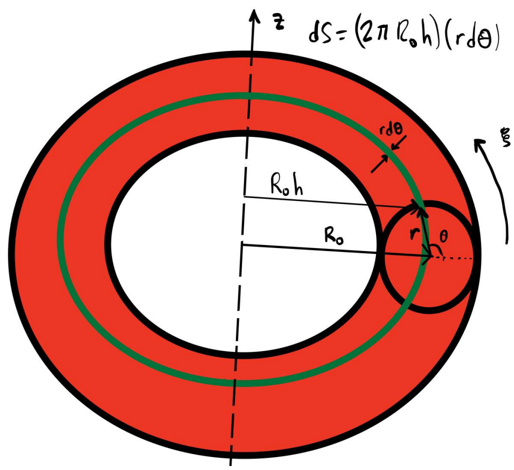

Figure 4: The green ribbon represents the infinitesimal area element dS used to calculate the flux-surface average $\langle A \rangle$. Remember, the green ribbon is at constant-θ.

the flux-surface average of A, we do

$$\langle A \rangle = \frac{1}{\int dS} \int A(r,\theta) dS = \frac{1}{\int h(r,\theta) d\theta} \int A(r,\theta) h(r,\theta) d\theta$$

where $h = 1 + \epsilon(r)\cos\theta$ and $\epsilon(r) = \frac{r}{R_0}$. Since $\int_0^{2\pi} \cos\theta d\theta = 0$, this reduces to

$$\langle A \rangle = \frac{1}{2\pi} \int_0^{2\pi} A(r,\theta) h(r,\theta) d\theta \tag{2.3}$$

With these tools in place, we are now ready to tackle transport in toroidal geometries.

2.3 Heuristic Estimate of Neoclassical Diffusion Coefficient

Todo: preview where we're going.

Curved magnetic field lines create particle drifts. In a tokamak without a poloidal magnetic field, the combination of the grad-B and curvature drifts leads to particles constantly drifting upwards (or downwards) in a tokamak and hence quickly out of the device. To prevent particles from drifting out of the device, a poloidal magnetic field is added to the machine by driving a toroidal plasma current. This means that as particles orbit around the tokamak, they change their position in θ in addition to in ζ. Although they are constantly drifting upwards (or downwards), the poloidal magnetic field creates an orbit in θ which is closed, or at least closed in the poloidal plane. This is illustrated in figure 5.

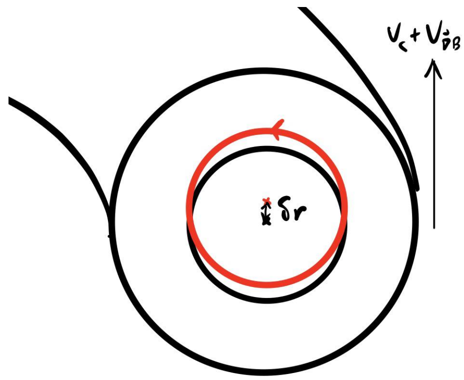

Figure 5: In a tokamak with a poloidal magnetic field (represented by the smaller black circle), the particles constantly drift upwards due to the curvature and grad-B drifts. Despite the fact that they are constantly drifting upwards, particles orbit in circles displaced vertically (represented by the red circle). In other words, the poloidal magnetic field stabilizes particle orbits.

Neoclassical diffusion is larger than classical diffusion because of the effect of curved magnetic field lines in toroidal geometries. As a particle drifts due to the curved field lines in a tokamak, it no longer orbits around the magnetic field line it started on. Instead, it will have some radial excursion from the radial flux surface it started on. In a tokamak with a poloidal magnetic field, the radial excursion will change with time (as the particle drifts radially), but after making a full poloidal orbit, the particle ends up at the same r it started on. However, suppose now that the particle collides at some point during it's orbit. Because the particle is drifting radially, the change in radial position of the guiding center Δx (relative to the r-position at $\theta = 0$) is equal to the radial excursion ξ at that point.[13] Sometimes that radial excursion will be positive and sometimes it will be negative. This randomness means that the radial position of the orbiting particles follows a random-walk diffusive process. The time between collisions Δt is the collision time, ν_{ei}^{-1}.

Let's try to estimate the step-size Δx, which is roughly the radial excursion ξ. and use that to estimate the neoclassical diffusion coefficient. Let's use the physical picture we developed earlier to help us figure out the step size. The two drifts we'd expect to see in a tokamak due to curved magnetic fields are the grad-B drift and the curvature drift.

$$\vec{v}_D = \frac{mv_\perp^2}{2qB^2}\hat{b} \times \vec{\nabla} B + \frac{mv_\parallel^2}{qB}\hat{b} \times (\hat{b} \cdot \vec{\nabla}\hat{b}) \tag{2.4}$$

[13]This statement ignores the physical reason for classical diffusion - particle collisions changing the guiding center position in a collision. For now, let's just estimate the diffusion coefficient due to radial excursions due to drifts. When we calculate the collisional neoclassical coefficient using MHD, both effects are automatically taken into account.

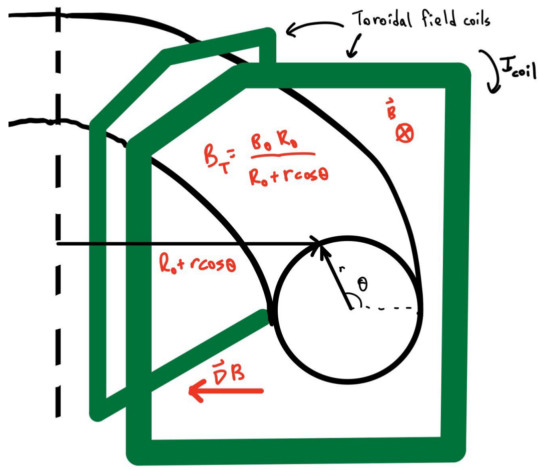

Figure 6: An illustration of the plasma and the toroidal field coils in a tokamak. The toroidal magnetic field goes like $1/(R_0 + r\cos\theta)$, which means that $\vec{\nabla} B$ points inwards towards the z-axis.

Let's think about what direction we expect these drifts to be in a tokamak. Since the magnetic field is primarily toroidal, and the toroidal magnetic field of a tokamak goes as $\frac{B_0}{h}$ (which we can calculate from Ampere's law), then the gradient of $\vec{B}$ points inwards, towards the z-axis of the tokamak. Thus, $\hat{b} \times \vec{\nabla} B$ points upwards, as we can see in figure 6. Similarly, the $\hat{b} \cdot \vec{\nabla}\hat{b}$ term points inwards along the radius of curvature, so $\hat{b} \times (\hat{b} \cdot \vec{\nabla}\hat{b})$ also points upwards. The radial component of the drift therefore goes like $|v_D| \sin\theta$, but for simplicity we'll just say

$$(v_D)_r \approx |v_D| \approx \frac{mV_T^2}{qBR_0}$$

Let's assume the plasma current and toroidal magnetic field are in the positive-ζ direction, so that the poloidal field points in the negative-θ direction and a drifting particle will drift upwards (or downwards, for negatively charged particles). Let's assume we have an orbiting positively-charged particle (with charge $q = e$) which has some positive v_ζ and starts at $\zeta = 0$, $\theta = 0$, and $r = r_0$. Thus, the particle starts on the outside of the tokamak. As it orbits, it's parallel velocity combined with the toroidal and poloidal magnetic field means that the particle initially streams along a field line towards the bottom of the tokamak. At the same time, the particle drifts up, off the field line it started on. While the particle is in the bottom half of the tokamak, the radial drift velocity is radially inwards. The radial excursion ξ at $\theta = -\pi/2$ is the maximum radial excursion ξ_{max}, so $r = r_0 - \xi_{max}$. Once the particle gets to $\theta = -\pi$ and begins orbiting in the upper half of the tokamak, the radial drift velocity becomes radially outwards. At $\theta = -3\pi/2 = \pi/2$, the particle reaches it's maximum radial excursion , so $r = r_0 + \xi_{max}$. As the particle orbits upwards towards $\theta = -2\pi = 0$ again, the radial excursion ξ decreases towards zero again. This is the same as the physical picture shown in figure 5. This physical picture implies that the strength of the poloidal magnetic field relative to the toroidal field impacts the radial excursion Δx. Let's calculate what we expect the maximum radial excursion to be.[14] Let's define q to be the number of times a magnetic field line goes around toroidally for each time the magnetic field line goes around poloidally. At zero poloidal field, $q = \infty$. As the poloidal magnetic field increases, q decreases. q is called the "safety factor". Based on our definition of q, we might imagine that

$$q = \frac{d\zeta}{d\theta} \tag{2.5}$$

Note that this agrees with our definition - if a magnetic field line goes toroidally around the tokamak 5 times for each time it goes around poloidally, then it goes 10π in ζ for every 2π in θ. This definition would give us a safety factor of 5, as we'd want. Now, let's try to figure out what the safety factor is in terms of the other variables. Suppose we travel some infinitesimal distance along the magnetic field line ds. The change in toroidal angle $d\zeta$ due to an infinitesimal distance along the field line will be proportional to the toroidal magnetic field B_t, and inversely proportional to the distance from the z-axis $R_0 h$. The change in poloidal angle $d\theta$ due to an infinitesimal distance traveled along the field line will be proportional to the poloidal magnetic field B_p, and inversely proportional to the distance from the major radius, r. Thus, our safety factor q is

$$q = \frac{B_t r}{B_p R_0 h}$$

For simplicity, we'll redefine the safety factor as (assuming $h \approx 1$)

$$q = \frac{B_t r}{B_p R_0} \tag{2.6}$$

Question: what exactly is the precise definition of the safety factor q?

We still haven't answered the question of what the radial excursion distance ξ_{max} is. Let's put everything together. The radial excursion ξ_{max} is approximately the radial drift velocity $(v_D)_r$ times the time it takes for the particle to go from zero to maximum excursion, or from $\theta = 0$ to $\theta = -\pi/2$. Well, the 'connection length' or the distance traveled by a particle to make this half-orbit is essentially the distance traveled toroidally, which is approximately $2\pi R_0 q/4$.[15]

[14] Remember, the radial excursion is roughly the step size Δx, which is what we're trying to calculate.

[15] Why is the connection length here $2\pi R_0 q/4$? Well, the 'connection length' is normally thought of as the distance a field line travels before it travels 2π in θ, 'connecting' back on itself. Since q is the number of toroidal orbits per poloidal orbit, then in a poloidal orbit a field line travels approximately q times the toroidal orbit distance, $2\pi R_0$. Here, however, we're only going $\pi/2$ in θ, so the connection length is 4 times smaller than the full connection length.

The typical parallel velocity is V_T, so the time it takes to go from minimum to maximum excursion is $\tau_{connection} \sim \frac{L}{v_\parallel} \sim \frac{\pi R_0 q}{2V_T}$. Thus, the radial excursion distance is

$$\xi_{max} \sim v_{Dr}\tau_{connection} \sim v_{Dr}\frac{\pi R_0 q}{2V_T} \sim \frac{mV_T q}{2eB} \tag{2.7}$$

We've estimated is the maximum radial excursion. However, the maximum radial excursion is also the step size Δx for neoclassical diffusion. This is because if the particle make a collision at a random point in it's orbit, then the difference in radial position between where it started and where it collided is somewhere between 0 and $2\xi_{max}$. Since we're just doing an estimate of the diffusion coefficient, we can just take this to be ξ_{max}. Remember, we're considering electrons. This means that

$$\Delta x \sim \xi_{max} \sim \frac{mV_{Te}q}{2eB} \sim \rho_e q$$

This gives a neoclassical diffusion coefficient of

$$D_{Neo} = (\Delta x)^2/\Delta t \sim \nu_{ei}\rho_e^2 q^2 = D_{Class}q^2 \tag{2.8}$$

Since typically $q \gg 1$ in a tokamak, this is much larger than the classical diffusion coefficient. Let's summarize what we've done, and make sure we understand why the collisional neoclassical diffusion coefficient is so much bigger than the classical diffusion coefficient. We recognized that in a realistic toroidal geometry, particles will orbit poloidally while drifting radially. The radial drift velocity, times the time it takes for particles to drift from zero to maximum excursion, gives the maximum radial excursion. The time it takes particles to drift between zero and maximum excursion is proportional to the connection length, which depends on the safety factor. We then assumed the particles guiding center changed by the approximately the maximum radial excursion between collisions, which gave us Δx and hence D.

Great, we've got an estimate for the collisional neoclassical diffusion coefficient, and we understand physically why it arises. If at this point, you still feel confused about collisional neoclassical diffusion, good. We'll have a lot more to say about collisional neoclassical diffusion in chapter 3, once we understand collisionless neoclassical diffusion. For now, let's calculate the collisional neoclassical coefficient more carefully using resistive MHD.

Question: Let's assume toroidal velocity v_ζ doesn't change in a collision. So even if a particle collides after drifting Δx in r, it's still going to be drifting up, which means it still forms a closed orbit. It seems like colliding won't change anything about the particle's orbit unless the toroidal velocity changes. Is that right?

2.4 Neoclassical Diffusion Coefficient: MHD

In chapter 1, we calculated the classical diffusion coefficient using the multi-fluid model as well as using resistive MHD. We found that the classical diffusion coefficient was $\rho_e^2\nu_{ei}$. Then, in section 2.3, we estimated the neoclassical diffusion coefficient by looking at the radial excursions of passing particles due to their drifts in curved magnetic fields. Now, we wish to calculate the neoclassical diffusion coefficient using resistive MHD. We'll see that up to a constant factor, we reproduce the results of our passing-particle estimate. Our strategy is going to be essentially the same as it was in chapter 1 when we calculated the classical diffusion coefficient using resistive MHD. However, the equations are going to be more complicated. We'll assume our torus is symmetric toroidally, but not poloidally. We'll then take a flux-surface average of the transport to get a flux-surface averaged diffusion coefficient. Our MHD equations are

$$\rho\frac{d\vec{u}}{dt} = -\vec{\nabla}P + \vec{J}\times\vec{B} \tag{2.9}$$

$$\vec{E} + \vec{u}\times\vec{B} = \eta\vec{J} \tag{2.10}$$

We assume our tokamak is in steady-state, and the velocities are sufficiently small that the convective derivative term is zero.[16] The pressure is assumed to depend only on r. We also know that the r-component of B must be zero, since we have only poloidal and toroidal magnetic field components (and $\vec{\nabla}\cdot\vec{B} = 0$). We assume that the magnetic field components, B_θ and B_ζ, can be written as

$$B_\theta = \frac{B_p(r)}{h} \tag{2.11}$$

$$B_\zeta = \frac{B_t(r)}{h} \tag{2.12}$$

You might be asking something along the lines of "but wait a second, isn't B_ζ already the toroidal magnetic field? And isn't B_θ already the poloidal magnetic field?" Well yes, that is true. But what we're doing is taking the θ-dependence out of $B_P(r)$ and $B_\theta(r)$, so that we have a quantity which is a constant over the flux-surface. With these assumptions, by components the MHD equations become

$$\frac{dP}{dr} = J_\zeta B_\theta - J_\theta B_\zeta \tag{2.13}$$

$$0 = J_r B_\theta \tag{2.14}$$

$$0 = J_r B_\zeta \tag{2.15}$$

For ohm's law, assuming $E_\zeta = 0$, we have[17]

$$u_\zeta B_\theta - u_\theta B_\zeta - \frac{\partial\phi}{\partial r} = \eta_\perp J_r \tag{2.16}$$

$$-u_r B_\theta = \eta_\parallel J_\zeta \tag{2.17}$$

$$u_r B_\zeta - \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \eta_\perp J_\theta \tag{2.18}$$

[16] Question: Is this why we set convective derivative to zero?

[17] Question: why is there no factor of h in the poloidal or radial electric fields? Depends on what $\vec{\nabla}A$ in toroidal coordinates is.

where $\eta = \frac{m_e \nu_{ei}}{ne^2}$ and apparently $\eta_\perp = \eta = 2\eta_\parallel$. We're going to try to solve for $\Gamma_r = nu_r$ in terms of $\frac{dn}{dr}$, where we'll use the constant temperature ideal gas law so that $\frac{dP}{dr} = 2k_B T \frac{dn}{dr}$. From equations 2.14 and 2.15, we know that $J_r = 0$. From equation 2.13, we have

$$J_\zeta = \frac{dP}{dr}\frac{1}{B_\theta} + J_\theta \frac{B_\zeta}{B_\theta} \tag{2.19}$$

In MHD, we have that $\vec{\nabla} \cdot \vec{J} = 0$. Since $J_r = 0$, and $\frac{\partial}{\partial \zeta} \to 0$, then (using equation 2.2 for the divergence in toroidal coordinates), we have

$$\vec{\nabla} \cdot \vec{J} = \frac{1}{rh}\frac{\partial}{\partial \theta}(hJ_\theta) = 0 \tag{2.20}$$

This implies that hJ_θ is not a function of θ, or

$$J_\theta = \frac{f(r)}{h} \tag{2.21}$$

If we can figure out $f(r)$, we'll be practically finished. We can just use equations 2.19 and 2.17 to solve for u_r, take the flux-surface average and multiply by n to get the neoclassical diffusion coefficient. This will of course take some work, but in principle if we can solve for $f(r)$ then we can figure our the diffusion coefficient. Isolating for $\frac{1}{r}\frac{\partial \phi}{\partial \theta}$ in equation 2.18, we have

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = u_r B_\zeta - \eta_\perp J_\theta \tag{2.22}$$

Using equation 2.17 to isolate for u_r, we have

$$u_r = -\frac{n_\parallel J_\zeta}{B_\theta} \tag{2.23}$$

Plugging u_r and J_θ into equation 2.22, we get

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\eta_\parallel J_\zeta B_\zeta}{B_\theta} - \eta_\perp \frac{f(r)}{h} \tag{2.24}$$

Now let's take the average over θ of this equation.[18] Since $\phi(0) = \phi(2\pi)$, then the LHS goes to zero.[19]

$$0 = -\frac{1}{2\pi}\int \left[\eta_\parallel \frac{B_\zeta}{B_\theta} J_\zeta + \eta_\perp \frac{f(r)}{h}\right] d\theta \tag{2.25}$$

Using equation 2.19, this becomes

$$0 = \frac{1}{2\pi}\int \left[\eta_\parallel \frac{B_\zeta}{B_\theta^2}\frac{dP}{dr} + \frac{f(r)}{h}(\eta_\perp + \eta_\parallel \frac{B_\zeta^2}{B_\theta^2})\right] d\theta \tag{2.26}$$

[18] Note that this is *not* the flux-surface average, as in equation 2.3. It's a simple average over θ.

[19] Note that the average of E_θ over θ is zero. Note that the flux-surface average of E_θ is not necessarily zero, and E_θ is not necessarily always zero.

Using equations 2.11 and 2.12, we can replace the B_ζ and B_θ with B_t and B_p.

$$0 = \frac{1}{2\pi}\int\left[\eta_\parallel h\frac{B_t}{B_p^2}\frac{dP}{dr} + \frac{f(r)}{h}(\eta_\perp + \eta_\parallel\frac{B_t^2}{B_p^2})\right]d\theta \tag{2.27}$$

Remember, B_p and B_t are only functions of r, while we'll assume that pressure is constant on each flux surface, so $\frac{dP}{dr}$ has no θ-dependence. Our integral over θ then only acts on $h(r,\theta)$. We have that[20]

$$\int hd\theta = \int(1+\epsilon\cos\theta)d\theta = 2\pi$$

$$\int\frac{1}{h}d\theta = \int\frac{1}{(1+\epsilon\cos\theta)}d\theta = \frac{2\pi}{\sqrt{1-\epsilon^2}}$$

so equation 2.27 reduces to

$$0 = \eta_\parallel\frac{B_t}{B_p^2}\frac{dP}{dr} + \frac{f(r)}{\sqrt{1-\epsilon^2}}(\eta_\perp + \eta_\parallel\frac{B_t^2}{B_p^2})$$

Solving for $f(r)$ gives

$$f(r) = \frac{-\eta_\parallel\frac{B_t}{B_p^2}}{(\eta_\perp + \eta_\parallel\frac{B_t^2}{B_p^2})}\sqrt{1-\epsilon^2}\frac{dP}{dr}$$

$$f(r) = -\frac{\sqrt{1-\epsilon^2}}{B_t}\left[\frac{1}{1+\frac{\eta_\perp B_p^2}{\eta_\parallel B_t^2}}\right]\frac{dP}{dr} \tag{2.28}$$

From equation 2.6, we have that $\frac{B_p}{B_t} = \frac{\epsilon}{q}$. In general, $q \gg 1$ and $\epsilon \ll 1$, so the term in the denominator is definitely small. We can therefore Taylor expand the denominator to get

$$f(r) \simeq -\frac{\sqrt{1-\epsilon^2}}{B_t}\frac{dP}{dr}(1 - \frac{\eta_\perp\epsilon^2}{\eta_\parallel q^2}) \tag{2.29}$$

Note that this is the negative of Professor Tangs's lecture notes, equation 1.67, because Bill uses a different sign convention for his toroidal coordinate system. With $f(r)$ in hand, we can solve for J_θ and J_ζ, using equations 2.21 and 2.19. Once we have J_ζ, we can solve for u_r using equation 2.23. We'll then take n times the flux-surface average $\langle u_r\rangle$ to get the diffusion coefficient.[21] Solving for J_ζ, we have

$$J_\zeta = \frac{dP}{dr}\frac{h}{B_p} - \frac{1}{h}\frac{B_t}{B_p}\left(\frac{\sqrt{1-\epsilon^2}}{B_t}\frac{dP}{dr}(1-\frac{\eta_\perp\epsilon^2}{\eta_\parallel q^2})\right)$$

[20] I'm not sure exactly how to do this second integral by hand, but that's not what we're worried about here.

[21] This assumes that n is not a function of θ, which isn't a great assumption. However, we have no way of knowing a priori what $n(\theta)$ will be, so for the purposes of calculating a diffusion coefficient it's fine just to assume n is not a function of θ.

$$J_\zeta = \frac{1}{B_p}\frac{dP}{dr}\left[h - \frac{1}{h}\sqrt{1-\epsilon^2}(1 - \frac{\eta_\perp \epsilon^2}{\eta_\parallel q^2})\right] \tag{2.30}$$

Thus, u_r becomes

$$u_r = -\frac{\eta_\parallel}{B_p^2}\frac{dP}{dr}\left[h^2 - \sqrt{1-\epsilon^2}(1 - \frac{\eta_\perp \epsilon^2}{\eta_\parallel q^2})\right] \tag{2.31}$$

Taking the flux surface average as in equation 2.3,[22] and using $\frac{1}{2\pi}\int h d\theta = 1$ and[23] $\frac{1}{2\pi}\int h^3 d\theta = 1 + \frac{3}{2}\epsilon^2$, and $\sqrt{1-\epsilon^2} \approx 1 - \frac{\epsilon^2}{2}$, we have (ignoring factors of ϵ^4)

$$\langle u_r \rangle = -\frac{\eta_\parallel}{B_p^2}\frac{dP}{dr}\left[1 + \frac{3}{2}\epsilon^2 - 1 + \frac{\epsilon^2}{2} + \frac{\eta_\perp \epsilon^2}{\eta_\parallel q^2}\right]$$

$$\langle u_r \rangle = -\frac{\eta_\parallel}{B_p^2}\frac{dP}{dr}\left[2\epsilon^2 + \frac{\eta_\perp \epsilon^2}{\eta_\parallel q^2}\right]$$

Using the ideal gas law, and setting $P = 2nk_BT$, this becomes

$$\langle u_r \rangle = -\frac{4\eta_\parallel \epsilon^2 k_B T}{B_p^2}\frac{dn}{dr}(1 + \frac{\eta_\perp}{2\eta_\parallel q^2}) \tag{2.32}$$

We can use a whole bunch of different results from before to simplify this expression. Firstly, remember that parallel resistivity is lower than perpendicular resistivity, so $\eta_\parallel = \frac{1}{2}\eta_\perp$. Secondly, remember that $\eta_\perp = \eta = \frac{m_e \nu_{ei}}{ne^2}$.[24] Also from the definition of q, we have that $B_p = B_t\frac{\epsilon}{q} \approx B\frac{\epsilon}{q}$. Putting these results together, we get

$$n\langle u_r \rangle = -\nu_{ei}\frac{2m_e k_B T}{e^2 B^2}(q^2+1)\frac{dn}{dr} = -\nu_{ei}\rho_e^2(1+q^2)\frac{dn}{dr} \tag{2.33}$$

Thus,

$$D_{Neo} = \nu_{ei}\rho_e^2(1+q^2) \tag{2.34}$$

This is the classical diffusion coefficient calculated using MHD, $\nu_{ei}\rho_e^2$, times a factor $(1+q^2)$. The first term represents the contribution due to classical diffusion, while the second term represents the collisional neoclassical diffusion coefficient. Since the safety factor q is generally much larger than 1 in fusion devices, the neoclassical diffusion coefficient is much larger than the classical diffusion coefficient. Note that within a factor of 2, this agrees with the heuristic model for the neoclassical diffusion coefficient we estimated in section 2.3. The heuristic model, which was based on single-particle drifts, gives essentially the same result as MHD, which is based on a highly-collisional fluid model.

Let's recap what just happened. We set out to calculate the neoclassical diffusion coefficient using MHD. Since MHD is a fluid model which assumes

[22]Remember, before we didn't take the flux-surface average, only the simple average over θ. *Now* we're taking the flux-surface average.

[23]Using Mathematica, admittedly. We could of course expand $(1+\epsilon\cos\theta)^3$ and perform a bunch of simpler integrals if we wanted to prove this.

[24]Question: What is going on here in the notes? Which is right?

high collisionality, then we expect this model to give us the right results in the limit where our plasma is very collisional.[25] We then solved the MHD equations in a toroidal geometry. To solve the equations, we had to find J_θ. To so do, we used $\vec{\nabla} \cdot \vec{J} = 0$, to find that $J_\theta = f(r)/h$. We were then able to solve for $f(r)$ by taking an average over θ to get rid of E_θ. With $f(r)$ in hand, we were able to solve for u_r. We then took the flux-surface average of u_r, multiplied by n, and used the ideal gas law to get our diffusion coefficient. Our neoclassical diffusion coefficient is much larger than the classical diffusion coefficient, by a factor $(1 + q^2)$. The physical reason for this increase in transport over the classical regime is the radial excursion of particles as they drift due to grad-B and curvature drifts.

In section 2.3, we estimated the neoclassical diffusion coefficient using the radial excursion of passing particles as they orbit poloidally around the tokamak. This is indeed the physical reason for the increase in D for the neoclassical regime. However, in order for the collisional fluid model to be accurate, these passing particles must (on average) collide multiple times per poloidal orbit. That means the condition on the validity of the collisional model is that the collision frequency ν_{ei} must be larger than the connection frequency, $\tau^{-1}_{connection}$.[26] This condition is

$$\frac{\nu_{ei}}{\tau^{-1}_{connection}} \sim \frac{\nu_{ei} R_0 q}{V_{Te}} \gg 1$$

2.5 Effects of non-zero E_ζ

Earlier in the chapter, we discussed how a poloidal magnetic field was necessary to prevent particles from drifting out of a tokamak. To create this poloidal magnetic field, a toroidal current is driven. To drive this toroidal current, we drive a toroidal electric field. To drive the toroidal electric field, we need a curl of $\vec{E}$ around the torus. By Faraday's law $\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$, to create a non-zero E_ζ we need a changing magnetic flux through the center of the tokamak. To do this, tokamaks have a giant solenoid at the center of the machine which continually ramps up the current,[27] changing the vertical magnetic field and creating the toroidal electric field E_ζ and hence the toroidal current.[28] This toroidal current also heats the plasma.

We can repeat the calculation of the previous section, including the possibility that there is a toroidal magnetic field E_ζ, and show that the radial particle

[25] We'll have more to say about when using the fluid model is justified in a moment.

[26] Remember, the connection time $\tau_{connection}$ is the time it takes for a particle to make a poloidal orbit, i.e. the time for a particle to follow a field line around 2π in θ. The word 'connection' is a bit of a misnomer because in general, field lines don't connect on themselves after going around 2π in θ.

[27] Question: does it ramp up? or does it ramp down?

[28] This process is one of the many reasons that we need to make a lot of progress before a steady-state tokamak is a realistic goal: the solenoid can't ramp up in current forever, which means we have a limit to the pulse length. It would be great if we could find a way to do steady-state current drive.

flux becomes

$$\Gamma_r = -(D_{Neo} + D_{Class})\frac{dn}{dr} - n_0\frac{E_\zeta}{B_p}\epsilon^2$$

This is one of the homework assignments, so I will not reproduce it in these notes. Typically, the toroidal electric field is quite small, only a few volts per $2\pi R$. In the homework, we then make a numerical estimate for this last term, and see that it is indeed very small relative to the other two terms, which implies that the toroidal electric field only negligibly changes the neoclassical transport in the collisional regime. In the collisionless regime, the electric field will become important.

2.6 Neoclassical Diffusion as a Result of Parallel Forces

Let's look at collisional neoclassical diffusion more generally. We'll look at the ζ-component of the multi-fluid equation, where a toroidal electric field is allowed. We'll see that the radial particle flux is due to a combination of forces parallel and perpendicular to the magnetic field. It turns out that the perpendicular forces are the forces which give rise to classical diffusion, while the parallel forces gives rise to neoclassical diffusion. Todo: do i want to say more?

We'll use the same toroidally-symmetric coordinate system we've been using, where we have a magnetic field in the toroidal direction $B_\zeta = \frac{B_T(r)}{h}$ and the poloidal direction $B_\theta = \frac{B_P(r)}{h}$. We'll also suppose we have a toroidal magnetic field E_ζ. We'll start with the multi-fluid momentum equation in a toroidal geometry.

$$m_\sigma n_\sigma \frac{d\vec{u}_\sigma}{dt} = q_\sigma n_\sigma(\vec{E} + \vec{u}_\sigma \times \vec{B}) - \vec{\nabla} P_\sigma + \sum_\alpha \vec{R}_{\sigma\alpha} \tag{2.35}$$

Since we're looking at diffusion, we can neglect the LHS of the momentum equation. We do this because (a) for diffusion, we are looking at steady state, so $\frac{\partial}{\partial t} \to 0$ and (b) since the only velocities are due to diffusion, we expect them to be small which allows us to neglect the $\vec{u}_\sigma \cdot \vec{\nabla}\vec{u}_\sigma$ term since it is second-order in velocity. Looking at the ζ-component of this equation, we have

$$q_\sigma n_\sigma(E_\zeta - u_r B_\theta) + \sum_\alpha R_{\zeta\sigma\alpha} = 0 \tag{2.36}$$

Now let's take the flux-average $\langle\rangle$ of this equation, where as usual flux-average means averaging over a constant-r surface.

$$\langle q_\sigma n_\sigma(E_\zeta - u_{r\sigma} B_\theta) + \sum_\alpha R_{\zeta\sigma\alpha}\rangle = 0 \tag{2.37}$$

$$q_\sigma\langle n_\sigma u_{r\sigma} B_\theta\rangle = \langle q_\sigma n_\sigma E_\zeta + \sum_\alpha R_{\zeta\sigma\alpha}\rangle$$

Remembering that the radial flux $\Gamma_{r\sigma} = \langle n_\sigma u_{r\sigma}\rangle$, we can solve for the radial flux in terms of the electric field and frictional force terms. Using $B_\theta = B_P(r)/h$

and $\langle \frac{1}{h} \rangle = \frac{1}{2\pi} \int \frac{1}{1+\epsilon\cos\theta} d\theta = \frac{1}{\sqrt{1-\epsilon^2}} \approx 1$ this becomes

$$q_\sigma B_P \langle n_\sigma u_{r\sigma} \rangle = \langle q_\sigma n_\sigma E_\zeta + \sum_\alpha R_{\zeta\sigma\alpha} \rangle$$

$$\Gamma_{r\sigma} = \langle n_\sigma u_{\sigma r} \rangle = \frac{1}{q_\sigma B_P} \langle \sum_\alpha R_{\zeta\sigma\alpha} + q_\sigma n_\sigma E_\zeta \rangle \tag{2.38}$$

We have, from $B_T \gg B_p$, that $\hat{b} \approx \hat{\zeta} + \frac{B_P}{B_0}\hat{\theta}$. We also have that $E_\zeta \approx E_\|$, where we've ignored $E_\perp$ because Question: why?. This means that[29]

$$R_{\zeta\sigma\alpha} = \hat{\zeta} \cdot \vec{R}_{\sigma\alpha} = \hat{\zeta} \cdot [\hat{b} R_{\|\sigma\alpha} + \vec{R}_{\perp\sigma\alpha}] \approx R_{\|\sigma\alpha} + \hat{\zeta} \cdot \vec{R}_{\perp\sigma\alpha} \tag{2.39}$$

Making these substitutions, the radial particle flux due to collisional neoclassical diffusion becomes

$$\Gamma_{r\sigma} = \frac{1}{q_\sigma B_P} \Big\langle [\sum_\alpha R_{\|\sigma\alpha} + n_\sigma q_\sigma E_\|] + \sum_\alpha \hat{\zeta} \cdot \vec{R}_{\perp\sigma\alpha} \Big\rangle \tag{2.40}$$

This is the result we wanted - we have the neoclassical collisional radial particle flux in terms of the forces parallel and perpendicular to the magnetic field.

Why are we interested in this result? Well, it tells us about the origins of the various types of diffusion. The second term, $\langle \sum_\alpha \hat{\zeta} \cdot \vec{R}_{\perp\sigma\alpha} \rangle$, gives us classical diffusion. This is the same forces which we found in chapter 1, where $\hat{\zeta}$ would be the z-axis. In the infinitely long cylinder from chapter 1, there would be no forces parallel to the cylinder. The first term is what gives us neoclassical diffusion.

But this general result tells us about the origins of the classical diffusion and neoclassical diffusion coefficients. The second term, $\frac{1}{q_\sigma B_P} \langle \sum_\alpha \hat{\zeta} \cdot \vec{R}_{\perp\sigma\alpha}$, gives us the classical diffusion coefficient. Classical diffusion in a fluid model (collisional regime) has to do with the perpendicular forces on particles. The first term in brackets gives us the neoclassical diffusion coefficient. The parallel forces on particles in the collisional regime give us the neoclassical diffusion.

Question: what is meant by $E_\|$ effects being balanced by $R_{\|ei}$?

Look back at PS diffusion equation - we have perp diffusion leading to classical diffusion. The parallel diffusion leads to classical diffusion.

Todo: this should probably be moved to chapter 2 somehow, right?

Todo: write about plateau too

Let's evaluate when the collisionless model of diffusion is valid: physically, we want a particle to be able to execute a bounce orbit before being scattered. If it is scattered before it completes a bounce orbit, then the radial excursion for the trapped particle is always going to be less than $\xi_{B\sigma}$, and the calculations we've done for the diffusion coefficient don't work. Mathematically, this condition comes out to

$$\nu_{eff} \ll \omega_B$$

[29] Question: I've ignored the factor of B_ζ/B_0, although Bill keeps it. I should ignore it, right?

or

$$\frac{\nu_{ei} q R_0}{V_T \epsilon^{\frac{3}{2}}} \ll 1 \tag{2.41}$$

If this is satisfied, then the neoclassical diffusion coefficient is approximately given by equation 3.28.

3 Collisionless Neoclassical Diffusion

"Just because Feynman says he is pro-nuclear power, isn't any argument at all worth paying attention to because I can tell you (for I know) that Feynman really doesn't know what he is talking about when he speaks of such things. He knows about other things (maybe). Don't pay attention to "authorities", think for yourself."

Richard Feynman

The previous chapter derived the neoclassical diffusion coefficient two separate ways. First, using a heuristic model for orbiting particles in a tokamak, we estimated the neoclassical diffusion coefficient based on the radial drifts of particles as they follow the twisted magnetic field lines in a tokamak, orbiting in θ. Second, we used resistive MHD to calculate the flux-surface averaged radial velocity u_r due to density gradients in a toroidal geometry. We then used the ideal gas law to get a neoclassical diffusion coefficient for a collisional plasma. You might find it surprising that these models gave us roughly the same estimate for the neoclassical diffusion coefficient. After all, one is a collisional fluid model, while the other is a model based on single-particle drifts. I certainly found it surprising. However, this really shouldn't be surprising, if we remember from GPP1 that summing the particle drift currents over species gives us the perpendicular component of the doubly-adiabatic MHD equations. The single-particle drift picture, accounting for the magnetization current, is equivalent to the fluid description of a plasma.

The analysis of the previous chapter, while important to understand, is ultimately wrong for a couple of reasons. The first is that in any real high-temperature plasma, the plasma is almost certainly *not* collisional, and therefore not in a Maxwellian distribution. For that reason, we should really be using a kinetic model rather than a fluid model to understand the diffusive transfer of heat and energy in a tokamak. The second is that in the single-particle drift model we used in the previous chapter, we didn't take into account the fact that some of the particles *aren't* passing particles, meaning that they don't make a full orbit in θ, but are reflected backwards by the gradient in B, like in a magnetic mirror. We call these particles 'banana particles' or 'trapped particles'. Trapped particles are a small fraction of the overall population in a tokamak, yet account for a disproportionately large amount of the overall transport. Essentially, the physical reason that the trapped-particle population is so important for transport is because they have such small $v_\parallel$. This means that they have much more time to drift radially, hence their radial excursion ξ_B[30] is much higher. It also means that they are more easily scattered in velocity space from trapped particles to passing particles.

You might be wondering why the fluid model doesn't take into account the

[30]The B stands for banana, so ξ_B is the radial excursion of the banana particles.

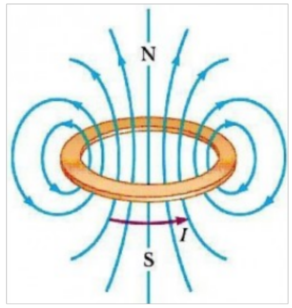

Figure 7: The magnetic field around a loop of wire. We can see that for this toroidal current, the poloidal magnetic field is stronger towards the center of the wire loop. Similarly, the poloidal magnetic field in a tokamak due to the toroidal field is stronger on the inside of the tokamak. This is why we write $B_\theta = \frac{B_p}{h}$.

trapped-particle population. After all, if we have a Maxwellian distribution of particles, some of those particles will of course be in the trapped particle population. We'll have more to say about this later in the chapter, but essentially the reason for this is the high-collisionality assumption of the fluid model. In the fluid model, those particles which are in the trapped-particle population collide multiple times before they execute a full banana orbit, and are quickly scattered from the trapped particle population to the passing particle population. In the fluid model, therefore, no single particle is able to orbit long enough without colliding to reach the full banana radial excursion ξ_B, even though at a given moment in time some fraction of the particles are in banana orbits. The fluid model takes into account the drift a typical particle has if it has many collisions during it's orbit.

3.1 Trapped Particles

In a tokamak, the magnetic field is strongest nearest to the z-axis, both for the toroidal and poloidal field components. The toroidal magnetic field goes like $1/h$ or $1/R$. We saw in figure 6 (back in chapter 2) that we can get this result from a simple Ampere's law calculation, taking $\int \vec{B} \cdot d\vec{l}$ around the torus. It turns out that not only is the toroidal magnetic field stronger nearer the z-axis, but the poloidal magnetic field is stronger there are well. We can visualize by looking at the magnetic fields due to a current loop like in figure 7. Although J_ζ is spread out over the volume of the tokamak and not carried in a wire, this figure captures the essence: there is a geometric enhancement of the poloidal magnetic field nearer the vertical axis due to a toroidal current in a tokamak. Unfortunately, we don't have a simple Ampere's law calculation to give us the poloidal magnetic field. Based on the above argument, we can see that following model of the magnetic field in a tokamak looks reasonable:

$$\vec{B} = \frac{1}{h}(B_t\hat{\zeta} + B_p\hat{\theta}) \tag{3.1}$$

where once again $h = 1+\epsilon\cos\theta$ and $\frac{B_p}{B_t} \ll 1$. Using this model, we can calculate the magnitude of B.

$$|B| = \frac{\sqrt{B_t^2 + B_p^2}}{(1+\epsilon\cos\theta)} \approx B_0(1-\epsilon\cos\theta)$$

The magnitude of the magnetic field depends on θ. The magnetic field is, for a given r, weakest at $\theta = 0$, and increases gradually until it reaches a maximum at $\theta = \pi$. Suppose we put a particle in our tokamak at $\theta = 0$, $r = r_0$, and $\zeta = 0$ with some velocity $v_\parallel$ and $v_\perp$, and watch what happens. Notice that, for a given r, we've inserted the particle at the minimum $|B|$. As our particle travels along the twisted magnetic field of the tokamak, it will change in θ, so the magnetic field must *increase* at it follows a field line until it reaches a maximum at $\theta = \pi$. What other situation in plasma physics does this remind you of? I'll give you a hint - for collisionless particles in slowly changing fields, we have the adiabatic invariant $\mu = \frac{mv_\perp^2}{2B}$. We also have constant-energy $E = \frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2}$. Perhaps you've recognized by now that a particle drifting in a tokamak is a lot like a particle drifting in a magnetic mirror machine. Unlike particles in a magnetic mirror, however, untrapped (i.e. passing) particles don't get lost outside of the device, but instead continually orbit poloidally. These untrapped particles are called passing particles. As we will show in section 3.1.1, the trapped particles are those particles which satisfy

$$\left(\frac{v_\parallel^2}{v_\perp^2}\right)\Big|_{B_{min}} \leq 2\epsilon \tag{3.2}$$

Since ϵ is a small number, this means that the trapped-particle population is generally a small fraction of the overall population, but increases as we increase in r.

3.1.1 Condition for Trapping

Let's derive this trapped particle condition now. We'll follow essentially the same steps we used to derive the trapping condition for a magnetic mirror. We have that

$$\frac{mv_\parallel^2}{2} + \mu B = E = \text{const}$$

For a particle which is just barely trapped, $v_\parallel$ becomes 0 at $B = B_{max}$, meaning all the energy is in perpendicular motion at B_{max}. This can be written as

$$\frac{m}{2}v_\parallel^2|_{B_{min}} + \frac{m}{2}v_\perp^2|_{B_{min}} \leq \frac{m}{2}v_\perp^2|_{B_{max}} = \mu B_{max}$$

$$\frac{m}{2}v_{\parallel}^2|_{B_{min}} + \frac{m}{2}v_{\perp}^2|_{B_{min}} \leq \left(\frac{m}{2B_{min}}v_{\perp}^2|_{B_{min}}\right)B_{max}$$

Dividing by $\frac{m}{2}v_{\perp}^2|_{B_{min}}$ and rearranging, we have

$$\left.\left(\frac{v_{\parallel}^2}{v_{\perp}^2}\right)\right|_{B_{min}} \leq \frac{B_{max}}{B_{min}} - 1 \tag{3.3}$$

Using $|B| \approx B_0(1 - \epsilon\cos\theta)$, we have

$$\frac{B_{max}}{B_{min}} - 1 = \frac{1+\epsilon}{1-\epsilon} - 1 \approx (1+\epsilon)(1+\epsilon) - 1 \approx 2\epsilon$$

Thus, the condition on trapping can be written as

$$\left.\left(\frac{v_{\parallel}^2}{v_{\perp}^2}\right)\right|_{B_{min}} \leq 2\epsilon \tag{3.4}$$

Great, so we've derived the trapping condition of equation 3.2. With a bit more algebra, we can write this condition a different way, which will be helpful to us later in this chapter. For barely trapped particles, $\frac{E}{\mu} = B_{max}$. For untrapped particles, $v_{\parallel}$ never goes to 0. Thus, $\frac{E}{\mu} > B_{max}$ for untrapped particles. For trapped particles, $\frac{E}{\mu} < B_{max}$. Actually, the lowest $\frac{E}{\mu}$ could be is B_{min}, which is when $v_{\parallel} = 0$ at B_{min}. This gives us, for trapped particles,

$$B_{min} < \frac{E}{\mu} < B_{max} \tag{3.5}$$

Now, let's define $\lambda = \frac{\mu B_0}{E}$. Dividing both sides of this expression by B_0, we have

$$1 - \epsilon < \frac{1}{\lambda} < 1 + \epsilon \tag{3.6}$$

Taking the inverse of this expression, we get

$$(1-\epsilon)^{-1} > \lambda > (1+\epsilon)^{-1}$$

For small ϵ, this becomes

$$1 + \epsilon > \lambda > 1 - \epsilon \tag{3.7}$$

This is another way of writing our condition for trapped particles, in addition to equation 3.2. We can see the conditions for trapped and untrapped particles summarized in table 1. Note that the intuition for $\lambda \equiv \frac{\mu B_0}{E}$ is that it is smaller when $v_{\parallel}$ is large relative to $v_{\perp}$ and larger when $v_{\perp}$ is large relative to $v_{\parallel}$. With this intuition in our minds, it should make sense that particles at larger λ should be trapped.

Take a look at figure 8. This figure is a bit tricky to figure out, so let's go through it. The solid line at the top represents the maximum λ allowed at a given $\epsilon = r/R_0$. The x-axis represents the angle θ, so the center is $\theta = 0$ while

Table 1: Conditions for Trapped and Untrapped particles in terms of $\lambda \equiv \frac{\mu B_0}{E}$

Trapped Partices	Untrapped particles
$1+\epsilon > \lambda > 1-\epsilon$	$1-\epsilon > \lambda > 0$

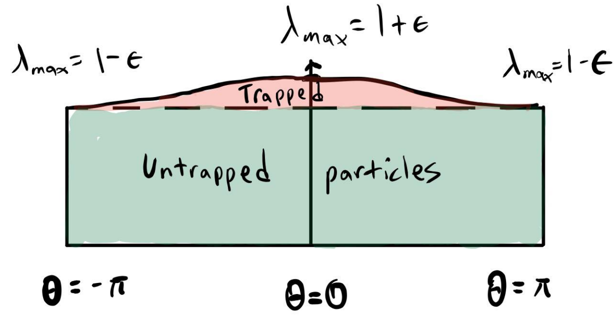

Figure 8: A plot of λ_{max} as a function of θ. The dashed line represents the function $\lambda = 1-\epsilon$, which is the dividing line between trapped and untrapped particles, as seen in table 1.

the edges are $\theta = \pi$. It makes sense that λ_{max} can be highest in the center, since $\lambda = \frac{\mu B_0}{E} = \frac{mv_\perp^2 B_0}{2BE}$ and B is lowest at $\theta = 0$. The particles with λ in the red region are the trapped particles, while those in the green region are the passing or untrapped particles. $\lambda = 1-\epsilon$ is the dividing line between trapped and untrapped particles, which is shown by the dashed line in this figure.

3.1.2 Fraction of Trapped Particles

Assuming an isotropic velocity distribution, we can show that the fraction of trapped particles is $\sqrt{1-\frac{B}{B_{max}}}$, while when we average this over a toroidal flux surface, the fraction of trapped particles becomes $\frac{4}{\pi}\sqrt{\frac{\epsilon}{2}}$. This is assigned as a homework problem, but it is important enough for understanding the physics that I will work it out in these notes. The fraction of trapped particles at any given point in our tokamak is given by

$$\sqrt{1-\frac{B}{B_{max}}} \tag{3.8}$$

Let's prove this. At an arbitrary point p in our tokamak, we have that $|B|_p = B_0(1-\epsilon\cos\theta)$. We also have that the particles which are just barely trapped at p are those who have zero parallel velocity at B_{max}. This trapping condition can be written as

$$\frac{m}{2}v_\parallel^2|_p + \mu B|_p \leq \mu B_{max}$$

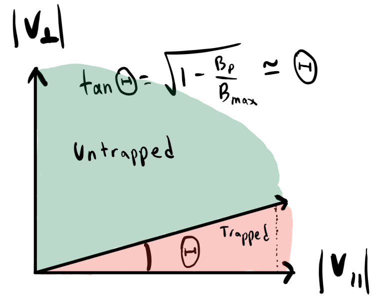

Figure 9: A plot of the trapped and untrapped particles in $|v_\parallel|$ and $|v_\perp|$ space.

Dividing by μB_p, we have

$$\left(\frac{v_\parallel^2}{v_\perp^2}\right)_p \leq \frac{B_{max}}{B|_p} - 1 \tag{3.9}$$

Note that our trapping condition 3.4 is just a special case of this equation, where we replace p with a point at $\theta = 0$. We can rewrite this as

$$\left|\frac{v_\parallel}{v_\perp}\right|_p \leq \sqrt{\frac{B_{max}}{B|_p}}\sqrt{1 - \frac{B|_p}{B_{max}}}$$

But $\frac{B_{max}}{B_p} = \frac{1+\epsilon}{1-\epsilon\cos\theta} \approx 1 + O(\epsilon)$ while $1 - \frac{B_p}{B_{max}} = 1 - \frac{1-\epsilon\cos\theta}{1+\epsilon} = O(\epsilon)$. We'll therefore ignore the first term in favor of 1, and keep the second term.

$$\left|\frac{v_\parallel}{v_\perp}\right|_p \leq \sqrt{1 - \frac{B_p}{B_{max}}} \tag{3.10}$$

We've plotted equation 3.10 in figure 9. If our velocity distribution is isotropic at each point in space, then each angle $d\Theta$ in velocity space is equally populated. Therefore the fraction of particles which are trapped is $\frac{\Theta}{\pi/2}$ where $\Theta \approx \tan\Theta = (1 - \frac{B}{B_{\max}})^{1/2}$. This means the fraction of trapped particles is

$$\frac{2}{\pi}(1 - \frac{B}{B_{max}})^{1/2}$$

This is the result Professor Tang cites, ignoring a factor of $\frac{2}{\pi}$. Question: This is different by a factor of $2/\pi$, right? When we average the number of trapped particles over a flux surface, we should find that the total fraction of trapped particles is $\frac{4}{\pi}\sqrt{\frac{\epsilon}{2}}$. Let's show this. Taking a flux-surface average, as we know, involves multiplying by $\frac{h}{2\pi}$ and integrating over θ, as in equation 2.3. We can write the fraction of trapped particles as

$$\frac{2}{\pi}\sqrt{1 - \frac{B}{B_{max}}} = \frac{2}{\pi}\sqrt{1 - \frac{1-\epsilon\cos\theta}{1+\epsilon}} \approx \frac{2}{\pi}\sqrt{\epsilon + \epsilon\cos\theta} = \frac{2}{\pi}\sqrt{\epsilon}\sqrt{1+\cos\theta}$$

Let's integrate this over a flux surface.

$$\frac{1}{\pi^2}\sqrt{\epsilon}\int_0^{2\pi}\sqrt{1+\cos\theta}(1+\epsilon\cos\theta)d\theta$$

We could do these integrals using trigonometric substitution, or simply plug them into Mathematica. Taking the second approach, we find that $\int_0^{2\pi}\sqrt{1+\cos\theta}d\theta = 4\sqrt{2}$ and $\epsilon\int_0^{2\pi}\sqrt{1+\cos\theta}\cos\theta d\theta = \frac{4\sqrt{2}}{3}\epsilon$. Since $\epsilon \ll 1$, to lowest order in ϵ the fraction of trapped particles are

$$\frac{4}{\pi^2}\sqrt{2\epsilon} \tag{3.11}$$

This is the same result Professor Tang gets in homework 3, except with the additional factor of $\frac{2}{\pi}$ which carries over from the previous calculation. Note that the important result is the $\epsilon^{1/2}$ dependence. The fraction of trapped particles is small at small ϵ but grows as r increases.

3.1.3 Effective Collision Frequency for Trapped Particles

The collision frequency for electron-ion collisions is

$$\nu_{ei} = \frac{4\pi n e^4}{m_e^2 V_{Te}^3}\ln\Lambda$$

The existence of an electron-ion collision frequency leads us to imagine that, on average, every ν_{ei}^{-1} seconds an electron collides with an ion and it's velocity at that moment is abruptly changed randomly. However, collisions in a plasma are a bit more complicated than that simple model. For a wonderful discussion of collisions in a plasma, see Bellan sections 1.8, 1.9, 1.10. Bellan writes "Grazing (small angle) collisions occur when the test particle impinges *outside* the shaded circle and so *occur much more frequently* than large angle collisions. Although each grazing collision does not scatter the test particle by much, there are far more grazing collisions than large angle collisions and so it is important to compare the *cumulative* effect of grazing collisions with the *cumulative* effect of large angle collisions." Bellan goes on to calculate that if there are many particles in a Debye sphere, then grazing collisions dominate large angle collisions. Let's investigate the diffusion of trapped particles in velocity space.

Looking at equation 1.5, we see that the mean-squared distance traveled due to a random-walk (diffusive) process $\langle x^2\rangle$ goes like Dt, where t is the time over which the random-walk occurred. This tells us that the time t it takes for a particle to, on average, diffuse a RMS distance x_{rms} is

$$t \sim \frac{1}{D}(x_{rms})^2$$

For random walks in angular displacement Θ, we can write this instead as

$$t \sim \frac{1}{D}(\Theta_{rms})^2 \tag{3.12}$$

How does this relate to the effective collision frequency of trapped particles? Why would we expect trapped particles to collide any differently than untrapped particles? Shouldn't their collision frequency be the same? This is all a bit tricky, so pay attention. Actually, yes, the collision frequency of trapped and untrapped particles is the same, in the sense that the collision diffusion coefficient D is the same for trapped and untrapped particles. There is not a different mechanism which causes collisions for trapped particles to be any different than for untrapped particles. *However*, what $(\nu_{ei})^{-1}$ really represents is the 90 degree scattering time, or the average time it takes for a particle to scatter 90 degrees in velocity space due to the cumulative effect of grazing collisions. For trapped particles, a particle only needs to diffuse roughly $\Theta \sim (2\epsilon)^{\frac{1}{2}}$ in velocity space[31] to be scattered out of the trapped particle population (as opposed to 90 degrees for a particle to be considered scattered when we normally consider collisions).

Let's look at equation 3.12. For normal scattering, the scattering time is $t = \nu^{-1}$ and the RMS angular displacement Θ_{rms} is $\frac{\pi}{2}$ radians. For scattering of trapped particles, the RMS angular displacement Θ_{rms} is $\sin(2\epsilon)^{\frac{1}{2}} \approx (2\epsilon)^{\frac{1}{2}}$, so the scattering time changes relative to the 90 degree scattering time by a factor

$$(\frac{(2\epsilon)^{\frac{1}{2}}}{\frac{\pi}{2}})^2 = \frac{8\epsilon}{\pi^2} \sim \epsilon$$

Since the scattering time is smaller by a factor of about ϵ, the scattering frequency is larger by a factor of roughly $1/\epsilon$. More precisely,

$$\nu_{eff\sigma} \sim \nu_{90\sigma}/\epsilon \tag{3.13}$$

where $\nu_{eff\sigma}$ is the effective scattering frequency for trapped particles of species σ due to collisions and $\nu_{90\sigma}$ is the effective 90 degree scattering frequency for particles of species σ.[32]

We can derive the effective collision frequency for trapped particles a different way, using the Lorentz collision operator. Remember, we derived the Lorentz collision operator from the Fokker-Planck equation back in section 1.4.1. This collision operator is

$$C_{coul}(f_e, f_i) = -\frac{\nu_{ei}}{2} V_{Te}^3 \frac{\partial}{\partial w_\alpha}\left[\left(\frac{\delta_{\alpha\beta}}{|w|} - \frac{w_\alpha w_\beta}{|w|^3}\right)\frac{\partial f_e}{\partial w_\beta}\right] \tag{3.14}$$

where $\vec{w} = \vec{v} - \vec{u}_i$. Now, it turns out that we can write the Lorentz collision operator using a different set of variables,

$$C_{coul}(f_e, f_i) = m_e \nu_{ei} \frac{v_\parallel}{B} \frac{\partial}{\partial \mu}\left[(v_\parallel \mu)\frac{\partial f_e}{\partial \mu}\right] \tag{3.15}$$

where $\mu = \frac{m_e v_\perp^2}{2B}$, $E = \frac{m_e v^2}{2}$, $B = B_0/h$, $v_\parallel = v\sqrt{1 - \frac{\lambda}{h}}$, $h = (1 + \epsilon\cos\theta)$, and $\lambda = \frac{\mu B_0}{E}$. Todo: prove this. Since $\lambda = \frac{\mu B_0}{E}$, then $\frac{\partial}{\partial \mu} = \frac{\partial}{\partial(\lambda E/B_0)} = \frac{B_0}{E}\frac{\partial}{\partial \lambda}$. Thus,

[31] To convince yourself of this, look at equation 3.4. When I say velocity space, really I mean velocity space at B_{min}, where $\theta = 0$.

[32] Typically, we just write this as ν_{ei} for electrons.

the Coulomb collision operator (equation 3.15) becomes

$$C_{coul}(f_e, f_i) = m_e \nu_{ei} \frac{v_\parallel B_0}{EB} \frac{\partial}{\partial \lambda}\left[(v_\parallel \mu \frac{B_0}{E}) \frac{\partial f_e}{\partial \lambda}\right]$$

Using $\xi = \frac{v_\parallel}{v} = \sqrt{1 - \frac{\lambda}{h}}$ for trapped particles, this becomes

$$C_{coul}(f_e, f_i) = 2\nu_{ei} \frac{h\xi}{v} \frac{\partial}{\partial \lambda}\left[(\xi v \lambda) \frac{\partial f_e}{\partial \lambda}\right]$$

$$C_{coul}(f_e, f_i) \sim \nu_{ei} \frac{h\xi^2 \lambda}{(\Delta\lambda)^2} f_e \tag{3.16}$$

The Coulomb operator is in general of order $\nu_{ei} f_e$. Let's estimate it's magnitude here. For trapped particles, we found earlier that the condition for trapping can be written as

$$1 - \epsilon < \lambda < 1 + \epsilon$$

Thus, for trapped particles $h \sim 1$, $\lambda \sim 1$, $\xi = \sqrt{1 - \frac{\lambda}{h}} \sim \sqrt{\epsilon}$, $\Delta\lambda \sim \epsilon$. Plugging these estimates into our estimate for the Coulomb operator, equation 3.16, we have

$$C_{coul}(f_e, f_i) \sim \frac{\nu_{ei}}{\epsilon} f_e \tag{3.17}$$

so the effective collision frequency for trapped particles is once again higher by a factor $\frac{1}{\epsilon}$.

Question: this seems to contradict the previous statement, where I said there wasn't a different mechanism for collisions with trapped particles and untrapped particles, it was just that they diffused more easily. But here the collision operator is actually different. What is going on here? Is there really a contradiction? What is the resolution?

3.1.4 Bounce Frequency of Trapped Particles

What does the bounce frequency of a trapped particle mean? In a mirror machine, we have particles bouncing between the two ends of a mirror, with a certain frequency. Similarly, in a tokamak, trapped particles are bouncing back and forth in θ, between θ_0 and $-\theta_0$, as in figure 10.

The bounce frequency is $\frac{2\pi}{\tau_B}$, where τ_B is the bounce period. Of course, these are just definitions, we'll need to calculate τ_B. Looking at figure 10, we can see that

$$\tau_B = \oint dt = 2\int_{-\theta_0}^{\theta_0} (\frac{d\theta}{dt})^{-1} d\theta = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\dot{\theta}}$$

To solve this, we need to relate $\dot{\theta}$ to $v_\parallel$. To do so we'll use our old friend, the safety factor q. Since q tells us roughly how many times a magnetic field line (and hence a particle) goes around toroidally for every time it goes around poloidally, then the toroidal distance a field line travels when it orbits 2π in

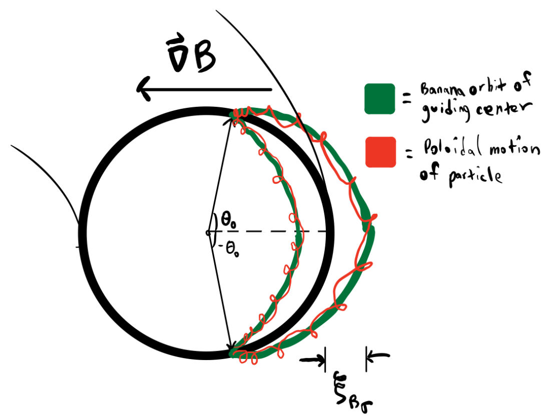

Figure 10: Trapped particles in a tokamak bounce between θ_0 and $-\theta_0$, while drifting radially outwards or inwards. The guiding center of these poloidal orbits (green) are shaped like bananas, so they are called banana orbits. Not shown is their toroidal motion.

poloidal angle is roughly $q2\pi R_0$.[33] This means that when a particle goes $d\theta$ in poloidal angle, the toroidal distance traveled $ds_\zeta \approx qR_0 d\theta$. Dividing by dt, we get

$$v_\zeta \approx qR_0\dot{\theta}$$

But for $q \gg 1$, $v_\zeta \sim v_\parallel$. Solving for $\dot{\theta}$,

$$\dot{\theta} = \frac{v_\parallel}{qR_0}$$

so

$$\tau_B = 2qR_0 \int_{-\theta_0}^{\theta_0} \frac{d\theta}{v_\parallel(\theta)} \tag{3.18}$$

Using conservation of magnetic moment and conservation of energy, we can solve for $v_\parallel(\theta)$ for trapped particles.

$$\frac{mv_\parallel(\theta)^2}{2} + \mu B(\theta) = E$$

Solving for $v_\parallel(\theta)$, we get

$$v_\parallel(\theta) = \Big(\frac{2}{m}\Big)^{1/2}\Big(E - \mu B(\theta)\Big)^{1/2}$$

Using $B(r,\theta) = \frac{B_0}{h} \approx B_0(1 - \epsilon\cos\theta)$ and $\lambda = \frac{\mu B_0}{E}$, this becomes

$$v_\parallel(\theta) = \Big(\frac{2E}{m}\Big)^{1/2}\Big(1 - \frac{\lambda}{h}\Big)^{1/2}$$

[33] Remember that this is the connection length.

$$v_\parallel(\theta) \approx \Big(\frac{2E}{m}\Big)^{1/2}\Big(1-\lambda(1-\epsilon\cos\theta)\Big)^{1/2} \tag{3.19}$$

So (using equation 3.18)

$$\tau_B = 2qR_0\sqrt{\frac{m}{2E}}\int_{-\theta_0}^{\theta_0}\Big(1-\lambda(1-\epsilon\cos\theta)\Big)^{-1/2}d\theta \tag{3.20}$$

I don't know how to solve this last integral exactly. If you're interested in how we do this, Bill does it in his notes. Instead, let's try to estimate the integral, using what we already know. We know that λ is a constant between $1-\epsilon$ and $1+\epsilon$ for trapped particles, so when we plug in λ into the expression $(1-\lambda(1-\epsilon\cos\theta))$, the factors of 1 will cancel and we'll be left with ϵ times some function of angle $f(\theta)$, plus terms to second order in ϵ which we'll ignore. We don't know what $f(\theta)$ is, but for simplicity we'll just say $\int_{-\theta_0}^{\theta_0} 1/\sqrt{f(\theta)}d\theta = O(1) \sim 1$. Writing all this out explicitly gives us

$$\int_{-\theta_0}^{\theta_0}\Big(1-\lambda(1-\epsilon\cos\theta)\Big)^{-1/2}d\theta \sim \int_{-\theta_0}^{\theta_0} 1/\sqrt{\epsilon f(\theta)}d\theta \sim \epsilon^{-1/2}O(1) \sim \epsilon^{-1/2}$$

With this simplification, we find

$$\tau_B \sim \frac{qR_0\sqrt{2m}}{(E\epsilon)^{\frac{1}{2}}} \sim \frac{qR_0}{V_{T\sigma}\epsilon^{\frac{1}{2}}} \tag{3.21}$$

$$\omega_B \sim \frac{V_{T\sigma}\epsilon^{\frac{1}{2}}}{qR_0} \tag{3.22}$$

This is our bounce frequency. Note that as we'd expect, electrons have a much higher bounce frequency than ions due to their higher thermal velocity. We also have that the bounce frequency goes like $\epsilon^{1/2}$, just like the fraction of trapped particles.

3.1.5 Banana Excursion Width

The banana excursion width $\xi_{B\sigma}$ is the radial excursion of a trapped particle during it's banana orbit. Take a look back at figure 10 to see what this looks like physically. We'll estimate the banana excursion width in two separate ways. The first is extremely simple: the radial drift velocity times the bounce period gives approximately the banana excursion width.[34] The second method will involve using the toroidal symmetry to apply conservation of canonical angular momentum p_ζ to solve for the excursion width.

[34]If you remember how we calculated the radial excursion distance ξ_{max} in the collisional regime back in chapter 2, we applied a very similar method. In chapter 2, however, we didn't multiply by the bounce period but rather multiply by a quarter of the connection length divided by the thermal velocity. I've used the same symbol ξ to remind us that these two quantities have similar physical origins.

The radial drift velocity in tokamaks due to the curvature and grad-B drifts, as we showed in section 2.3, point either upwards or downwards depending on the sign of the particle. As we showed, the radial drifts have magnitude

$$|v_{Dr}| \sim \frac{mV_{T\sigma}^2}{eBR_0}$$

Therefore, an approximate estimate of the banana excursion $\xi_{B\sigma}$ is

$$\xi_{B\sigma} \sim v_{Dr}\tau_B \sim \frac{mV_{T\sigma}^2}{eBR_0}\frac{qR_0}{V_{T\sigma}\epsilon^{\frac{1}{2}}} \sim \frac{mV_{T\sigma}q}{eB\epsilon^{\frac{1}{2}}} \sim \frac{\rho_\sigma q}{\epsilon^{\frac{1}{2}}} \tag{3.23}$$

Since $q \gg 1$ and $\epsilon \ll 1$, the banana excursion width is much larger than the gyroradius ρ_σ. Next let's calculate the banana excursion width a second way, using conservation of angular momentum. For systems which are symmetric in ζ, we have conservation of p_ζ for electrons,

$$p_\zeta = m_e R v_\zeta - eRA_\zeta = \text{const}$$

Here is the key: using $B = \vec{\nabla} \times \vec{A}$ and $\frac{\partial}{\partial \zeta} \to 0$, we know that $B_\theta = -\frac{\partial A_\zeta}{\partial r}$. So (Question: but doesn't this ignore a factor of h? where does h show up in the curl?)

$$A_\zeta = -\int_0^r B_\theta(r')dr'$$

Suppose the mean radius (center) of the banana orbit is r_0.[35] We can then expand A_ζ to first order in a Taylor series around $r = r_0$.

$$A_\zeta = -\int_0^{r_0} B_\theta(r')dr' - (r - r_0)B_\theta(r_0)$$

Now, at the turning point ($\theta = \theta_0$), we have that $v_\parallel \approx v_\zeta = 0$, and $r - r_0 = 0$.[36] Thus, at $\theta = \theta_0$,

$$p_\zeta = eR\int_0^{r_0} B_\theta(r')dr' \tag{3.24}$$

At $\theta = 0$, we have $r - r_0 = \xi_{Be}$. So at $\theta = 0$,

$$p_\zeta = eR\int_0^{r_0} B_\theta(r')dr' + mRv_\zeta + eR\xi_{Be}B_\theta(r_0) \tag{3.25}$$

From constancy of p_ζ, we can set equations 3.24 and 3.25 equal, giving

$$eR\xi_{Be}B_\theta(r_0) = -mRv_\zeta$$

[35] r_0 would be the radius of the black circle in figure 10.

[36] This highlights a subtle difference in our treatment of the passing particles and the trapped particles. For the passing particles, we set the center of their orbits to be at $\theta = 0$, while for trapped particles we're setting the center of their orbit at $\theta = \theta_0$ or $-\theta_0$.

Using equation 2.6 to show that

$$B_\theta \approx B_p \approx \frac{B_T\epsilon}{q} \approx \frac{B\epsilon}{q}$$

we see that the banana excursion width at $\theta = 0$, which is the maximum banana excursion width, is

$$\xi_{Be} = -\frac{mv_\zeta}{eB_\theta} \approx -\frac{mv_\zeta q}{eB\epsilon}$$

Now, naively we would think to plug in $v_\zeta \sim v_\parallel \sim V_T$. However, this is wrong. Can you see why? The reason is because v_ζ is evaluated at $\theta = 0$ for the *trapped particles*. The untrapped particles have $v_\parallel \sim V_T$. However, the trapped particles have most of their energy in the perpendicular motion, and very little in the parallel motion. Using the trapping condition

$$\left(\frac{v_\parallel}{v_\perp}\right)_{B_{min}} \leq \sqrt{2\epsilon}$$

we have that $v_\zeta \sim v_\parallel \sim v_\perp\sqrt{\epsilon} \sim V_{Te}\sqrt{\epsilon}$. Thus, our banana excursion width can be written as

$$\xi_{Be} = -\frac{mv_\zeta q}{eB\epsilon} = -\frac{\rho_e q}{\epsilon^{\frac{1}{2}}} \tag{3.26}$$

This is the same[37] as what we estimated for the banana excursion width using simply $\xi_{B\sigma} \sim v_{Dr}\tau_B$.

3.1.6 Banana Diffusion Coefficient

We have all the tools in place to estimate the banana diffusion coefficient. We know the approximate random-walk step size Δx for trapped particles - the banana excursion width, ξ_{Be}.[38] We know the approximate time for a particle to become untrapped: $\Delta t \sim \nu_{eff}^{-1}$. Naively, this would give us a banana diffusion coefficient $\xi_{Be}^2\nu_{eff}$. However, this is wrong. The reason this is wrong is that only a fraction of the total number of partices are actually in banana orbits. This fraction, which we calculated in section 3.1.2, is of order $\epsilon^{\frac{1}{2}}$. To get the modified diffusion coefficient, we simply multiply the diffusion coefficient of trapped particles by the fraction of particles which are trapped.[39] Therefore, our banana diffusion coefficient is

$$D_{Ban} \sim \langle f_T\rangle \frac{(\Delta x)^2}{(\Delta t)} \sim \epsilon^{\frac{1}{2}} \frac{\rho_e^2 q^2}{\epsilon} \frac{\nu_{ei}}{\epsilon} \tag{3.27}$$

---

[37] Except for an unimportant minus sign, which we got because we assumed v_ζ was positive.

[38] Make sure you understand physically why this is true. The banana particles, when scattered, change their radial position relative to their most recent scattering by roughly their radial excursion. The amount the particles change their radial position per ‘step’ is the radial step-size.

[39] If you want to see why this is true, look at equation 1.1. If there is a density gradient in n, but only a fraction of the particles which make up that n experience diffusion, then the flux will be smaller by this fraction. Therefore the effective diffusion coefficient must also be smaller by the fraction of particles which actually diffuse.

$$D_{Ban} \sim \rho_e^2 \nu_{ei} \frac{q^2}{\epsilon^{\frac{3}{2}}} \tag{3.28}$$

This banana diffusion coefficient is larger than the collisional (fluid) neoclassical diffusion coefficient we calculated back in chapter 2. In fact, it's larger by a factor of $\epsilon^{-3/2}$, which is much larger than 1. Since the collisional neoclassical diffusion coefficient is already much larger than the classical diffusion coefficient, we can see that collisionless neoclassical diffusion is indeed quite large.

Why is the collisionless neoclassical diffusion coefficient so much larger than the collisional neoclassical diffusion coefficient? Essentially, it's because the collisional model doesn't account for the trapped particles. These trapped particles, while a small fraction of the particles, account for a large fraction of the diffusion. Why do they diffuse so quickly? It comes down to (a) the fact that their parallel velocity is small, so their radial excursion is much larger, and (b) they only need to scatter a small amount in velocity space to no longer be trapped particles, meaning their effective collision frequency is much larger.

3.1.7 Neoclassical Diffusion Regimes

Let's recap. The trapped particles are those particles which don't have enough $v_\parallel$ to circulate around the tokamak due to μ-conservation and the magnetic field going as $\sim 1/h$. These particles are those particles which, at B_{min}, have

$$\left(\frac{v_\parallel^2}{v_\perp^2}\right)_{B_{min} \leq 2\epsilon}$$

The flux-surface averaged fraction of trapped particles is

$$f_T = \frac{4\sqrt{2}}{\pi^2}\sqrt{\epsilon}$$

Since the trapped particles only need to scatter a small angle in velocity space to become untrapped, the effective collision frequency for trapped particles is higher than the normal collision frequency.

$$\nu_{eff,e} \sim \nu_{ei}/\epsilon$$

The bounce frequency of trapped particles is, integrating $v_\parallel$ from one turning point to another,

$$\omega_B \sim \frac{V_{T\sigma}\epsilon^{1/2}}{qR_0}$$

The banana excursion width is

$$\xi_{B\sigma} = \frac{\rho_\sigma q}{\epsilon^{1/2}}$$

Using the banana excursion width as Δx and the inverse effective collision frequency as Δt, the banana diffusion coefficient is

$$D_{Ban} \sim \rho_e^2 \nu_{ei} \frac{q^2}{\epsilon^{3/2}}$$

The physical origin of collisional neoclassical diffusion is similar, but because the trapped particles collide before they can execute a banana orbit, then we're interested in the radial excursions of passing particles. The passing particles have a radial drift velocity of order

$$v_{Dr} \approx \frac{m_\sigma V_{T\sigma}^2}{qBR_0}$$

The connection length is the distance traveled by a field line before it ends up at the same θ it began with.

$$L_{connection} \approx 2\pi R_0 q \sim R_0 q$$

The connection time is the time it takes for a typical particle to make a complete poloidal orbit as it follows a field line.

$$\tau_{connection} \sim L_{connection}/V_{T\sigma} \sim R_0 q/V_{T\sigma}$$

The radial excursion for passing particles is

$$\xi_{max} \sim \frac{mV_{T\sigma}q}{2eB}$$

Using the radial excursion as the random-walk step size Δx, and the inverse collision frequency as the time between random-walk steps Δt, we get the collisional neoclassical diffusion coefficient D_{Neo}

$$D_{Neo} \sim \nu_{ei}\rho_e^2 q^2 \sim D_{Class} q^2$$

Great, we've summarized many of the important results from collisional and collisionless neoclassical transport theory.

What we'd like to do now is understand how the collision frequency, relative to the bounce frequency, can be used to tell us which of the neoclassical transport regimes we are in. As we'll see, there are actually three (not two!) collisional regimes. The least-collisional regime is where the trapped particles are able to execute bounce orbits without colliding. This is called the *banana regime*. As the collision frequency increases, eventually the trapped particles are no longer able to execute bounce orbits without colliding, but the untrapped particles are still able to make full poloidal orbits without colliding. This is called the *Plateau regime*, for reasons we will soon understand. At still higher collisionality, the untrapped particles are not able to make full poloidal orbits before colliding. This is called the *Pfirsch-Schluter (P-S) regime*, or the fluid-like regime. In evaluating which of these three regimes we're in, the dimensionless parameter we want to consider is ν_e^*, the ratio of the effective collision frequency for trapped particles and the bounce frequency of trapped particles.

$$\nu_e^* = \frac{\nu_{eff,e}}{\omega_B} = \frac{\nu_{ei} q R_0}{V_{Te}\epsilon^{3/2}} \tag{3.29}$$

To be in the banana regime, we need $\nu_e^* \ll 1$, so that trapped particles bounce many times before becoming untrapped.

What about particles in P-S regime? Since the passing particles collide before making a poloidal orbit, then in this regime

$$\frac{\nu_{ei}}{\tau_{connection}^{-1}} \sim \frac{\nu_{ei} R_0 q}{V_{Te}} = \nu_e^* \epsilon^{3/2} \gg 1$$

$$\nu_e^* \gg \epsilon^{-3/2} \tag{3.30}$$

Remember, since $\epsilon \ll 1$, then the normalized collision frequency ν_e^* in the fluid-like (P-S) regime is much greater than 1. This tells us that between $\nu_e^* = 1$ and $\nu_e^* = \epsilon^{-3/2}$, we have the plateau regime. Remember, in the plateau regime, trapped particles become untrapped before they make a complete banana orbit, but passing particles are able to make a full orbit in θ before they collide.

Let's look at the diffusion coefficients as a function of ν_e^*.[40] In the collisionless banana regime, we have

$$D_{Ban} \sim \rho_e^2 \nu_{ei} \frac{q^2}{\epsilon^{3/2}}$$

and

$$\nu_{ei} = \frac{\nu_e^* V_{Te} \epsilon^{3/2}}{q R_0}$$

which gives us

$$D_{Ban} \sim \rho_e^2 V_{Te} \frac{q}{R_0} \nu_e^* \tag{3.31}$$

Similarly, in the highly collisional P-S regime, we have

$$D_{Neo} \sim \nu_{ei} \rho_e^2 q^2$$

which gives us

$$D_{Neo} \sim \rho_e^2 V_{Te} \frac{q}{R_0} \nu_e^* \epsilon^{3/2} \tag{3.32}$$

Now check this out. At the transition from the collisionless regime to the plateau regime, we have $\nu_e^* = 1$, which (from equation 3.31) gives $D = \rho_e^2 V_{Te} \frac{q}{R_0}$. At the transition from the plateau regime to the P-S regime, we have $\nu_e^* = \epsilon^{-3/2}$, which (from equation 3.32) gives us $D = \rho_e^2 V_{Te} \frac{q}{R_0}$. The conclusion is obvious: the neoclassical diffusion coefficient is the same at the low-collisionality end of the plateau regime as it is at the high-collisionality end of the plateau regime. This tells us that the diffusion coefficient $D_{Plateau}$ is constant throughout the plateau regime.

$$D_{Plateau} = \rho_e^2 V_{Te} \frac{q}{R_0} \tag{3.33}$$

[40] I need to make an important comment here: we've calculated D in a highly collisional regime, and in the collisionless regime. We haven't calculated D in the intermediate 'plateau regime', where the banana particles don't make complete orbits but the passing particles do. We're going to estimate D in the plateau regime now.

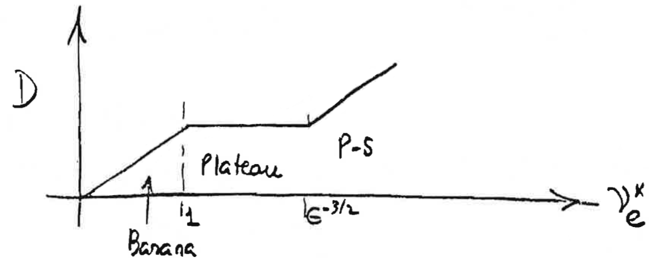

Figure 11: A plot of the diffusion coefficient as a function of the bounce-normalized collision frequency ν_e^*. We've previously calculated the diffusion coefficient for the banana and P-S regimes. We see that the diffusion coefficient is the same at the beginning and end of the plateau regime, which motivates us to believe that the diffusion coefficient doesn't change as a function of ν_e^* in the plateau regime.

Equations 3.31, 3.32, and 3.33 are plotted as a function of ν_e^* in figure 11. Make sure you understand what this plot is telling us. The diffusion coefficient is largest in the high-collisionality regime, stays constant throughout the plateau regime, and is lowest in the low-collisionality regime. Despite this fact, since $D \propto \nu_{ei}$, then the collisionless diffusion coefficient is larger *for a given* ν_{ei}.[41]

3.2 Electric Field and Trapped Particles

Before we look at the Ware pinch, let's look at what we've done so far in a slightly different way.

Todo: explain what ware pinch is

3.2.1 Heuristic Estimate of Ware Pinch

A parallel electric field in the collisionless regime leads to a very different effect, called the Ware pinch. Todo: explain Ware pinch

Let's estimate the average radial velocity of a trapped electron in a tokamak. We start with the flux function ψ for a toroidal geometry. We define

$$\psi \equiv \int RB_\theta dr \tag{3.34}$$

Note also that

$$\psi = \int R_0 h \frac{B_P}{h} dr = \int R_0 B_P dr \tag{3.35}$$

The flux function ψ has the intepretation of the poloidal flux (or more precisely, θ-flux) enclosed within a constant-r surface. It tells us how much our

[41] Question: on page 38, it says plateau regime is very... what?

poloidal field changes as a function of r, since

$$\frac{\partial\psi}{\partial r} = RB_\theta = R_0 B_P \tag{3.36}$$

We have another fact about ψ to keep in mind: Since $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\frac{\partial}{\partial\zeta} \to 0$ in a toroidal geometry, we have $B_\theta = -\frac{\partial}{\partial r}A_\zeta$. This means that we have another useful form for ψ,

$$\psi = -RA_\zeta \tag{3.37}$$

Note that $\frac{\partial\psi}{\partial\theta} = 0$, since (using equation 3.35)

$$\frac{\partial\psi}{\partial\theta} = \int \frac{\partial}{\partial\theta} R_0 B_P(r) dr = 0$$

This means that ψ is constant on a flux surface of constant-r. [42] Since any electric field in a tokamak is being driven by electromagnetic induction, then we have $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{dt}$. The θ-component of this is

$$-\frac{\partial}{\partial r}E_\zeta = -\frac{\partial}{\partial t}B_\theta$$

This is where the fact that we are *estimating* the average radial drift velocity becomes important. Question: Do I have this right? For small ϵ, $R \approx R_0$. This means that $B_\theta = \frac{1}{R}\frac{\partial\psi}{\partial r} \approx \frac{\partial}{\partial r}\frac{\psi}{R_0}$. We can therefore remove the derivative with respect to r in the above equation, and write

$$\frac{\partial\psi}{\partial t} = E_\zeta R_0 \approx E_\zeta R \tag{3.38}$$

We've got some nice facts about this flux function ψ. Let's start thinking about the radial velocity of a particle. We can write the radial velocity of a particle in a funny way, which will help us later on.

$$v_r = \frac{\vec{v}\cdot\vec{\nabla}\psi}{B_\theta R} \tag{3.39}$$

Now let's look at conservation of angular momentum in the toroidal direction ζ. We have (using equation 3.37)

$$p_\zeta = mRv_\zeta - eRA_\zeta = \text{Const} \tag{3.40}$$

$$\psi = -RA_\zeta = \text{Const} - \frac{m}{e}Rv_\zeta \tag{3.41}$$

$$\frac{d\psi}{dt} = -\frac{mR}{e}\frac{d}{dt}v_\zeta \tag{3.42}$$

[42]Notice that constant RA_ζ is the definition used in GPP1 for a flux-surface in a cylindrically symmetric geometry. These thing are the same!

In the last step, notice that the time-derivative is a total derivative, not a partial derivative. We are taking a total derivative because we are going to be looking at the convective derivative of a particle as it travels around a banana orbit. Next, we look at a trapped particle and integrate equation 3.42 over time between the two turning points of the banana orbit. This gives us

$$-\frac{mR}{e}\int \frac{dv_\zeta}{dt}dt = \int dt[\frac{\partial \psi}{\partial t} + \vec{v}\cdot\vec{\nabla}\psi]dt$$

$$0 = \int \frac{ds}{v_\parallel}[\frac{\partial \psi}{\partial t} + \vec{v}\cdot\vec{\nabla}\psi]dt$$

Where we've used the fact that $v_\parallel \propto v_\zeta = 0$ at the turning points to set the LHS equal to zero. ds is the infinitesimal displacement along the field line. We can rewrite this as

$$\overline{\frac{\partial \psi}{\partial t}} = -\overline{\vec{v}\cdot\vec{\nabla}\psi} \tag{3.43}$$

where the bar stands for averaging in time over a banana orbit. Hey wait, the term on the RHS looks like our funny expression for the radial velocity in equation 3.39! Let's plug that in and see what happens.

$$\overline{\frac{\partial \psi}{\partial t}} = \overline{v_r R B_\theta}$$

From equation 3.42, we have

$$v_r \approx \frac{E_\zeta}{B_\theta} \tag{3.44}$$

Todo: explain why this isn't just the E times B velocity. Explain why this is actually inwards due to my sign convention

Todo: conclusion

3.3 Kinetic Analysis

We're deriving the Lorentz conductivity of a plasma. This was done in GPP1, we're doing this a different way.

We start with the steady-state drift-kinetic equation for electrons

$$(v_\parallel \hat{b} + \vec{v}_D)\cdot\vec{\nabla}f + \frac{e}{m}E_\parallel v_\parallel \frac{\partial}{\partial \mathcal{E}}f = C(f) \tag{3.45}$$

Question: why not -e?

How did we get this? Well we started with the Vlasov-Maxwell equation for electrons, and assumed steady-state so $\frac{\partial}{\partial t} \to 0$. The only perpendicular velocity of the particles, if they are drifting, is v_D (question: why?). The $\frac{q}{m}(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{\nabla}_v$ term in the Vlasov-Maxwell equation has been modified by ignoring $\vec{B}$ and using $\mathcal{E} = \frac{1}{2}mv^2$ to get

$$\frac{\partial}{\partial \vec{v}} = \frac{\partial \mathcal{E}}{\partial \vec{v}}\frac{\partial}{\partial \mathcal{E}} = (mv_\parallel \hat{b} + m\vec{v}_\perp)\frac{\partial}{\partial \mathcal{E}}$$

$$\frac{q}{m}\vec{E}\cdot\vec{\nabla}_v f = qE_\parallel v_\parallel \frac{\partial}{\partial \mathcal{E}} f$$

The collision operator is the Lorentz Collision operator, which gives 0 when operating on a Maxwellian distribution. The crucial step is to linearize around a zero-field, equilibrium distribution and assume the electric field creates a perturbed f which is small relative to the equilibrium Maxwellian distribution. This will be true if (?). This means that $f = f_0 + f_1 + ...$ where $f_0 = f_M$, and $C(f_0) = 0$. We linearize the drift-kinetic equation around this equilibrium, where v_D and $E_\parallel$ are both first-order quantities.

$$v_\parallel \hat{b}\cdot\vec{\nabla} f_1 - C(f_1) = -\vec{v}_D\cdot\vec{\nabla} f_0 - ev_\parallel E_\parallel \frac{\partial}{\partial \mathcal{E}} f_0$$

Since

$$f_M = n\Big(\frac{m}{2\pi k_B T}\Big)^{3/2}\exp\Big(-\frac{mv^2}{2k_B T}\Big) \tag{3.46}$$

then

$$\frac{\partial}{\partial \mathcal{E}} f_M = -\frac{f_M}{k_B T} \tag{3.47}$$

this becomes

$$v_\parallel \hat{b}\cdot\vec{\nabla} f_1 - C(f_1) = -\vec{v}_D\cdot\vec{\nabla} f_0 - \frac{e}{k_B T} v_\parallel E_\parallel f_0 \tag{3.48}$$

Now, we are looking for the contribution to the current due to the trapped-particle population or something like that. To isolate for this, we can separate out the contribution from the classical Spitzer conductivity from that of the bootstrap component. Thus, we define the Spitzer conductivity contribution to the first-order perturbation to f, f_S, as the solution to the equation

$$C(f_S) = \frac{e}{k_B T} v_\parallel E_\parallel f_M$$

Let's now try to solve for the contribution from the bootstrap current in a toroidal geometry, by setting $f_1 = f_S + \hat{f}$.

$$v_\parallel \hat{b}\cdot\vec{\nabla}\hat{f} + v_\parallel \hat{b}\cdot\vec{\nabla} f_S - C(\hat{f}) - C(f_S) = -\vec{v}_D\cdot\vec{\nabla} f_0 - \frac{e}{k_B T} v_\parallel E_\parallel f_0$$

If we assume that the quantities n and T in the Maxwellian f_M only vary with poloidal radius r, then $\vec{\nabla} f_M = \frac{\partial f_M}{\partial r}$, so (using equation 3.46)

$$\vec{\nabla} f_0 = \frac{f_M}{n(r)}\frac{\partial n}{\partial r} - \frac{3f_M}{nT(r)}\frac{\partial T}{\partial r} + f_M\frac{mv^2}{2k_B T^2}\frac{\partial T}{\partial r}$$

$$\vec{\nabla} f_0 = [\frac{\partial(\ln n_0)}{\partial r} + (\frac{v^2}{V_T^2} - \frac{3}{2})\frac{\partial \ln T}{\partial r}] f_M \tag{3.49}$$

where $V_T^2 = \frac{2k_BT}{m}$. Now we subtract equation 3.47 out of equation 3.48 and use equation 3.49.

$$v_\parallel \hat{b} \cdot \vec{\nabla} \hat{f} - C(\hat{f}) = -(v_D)_r \Big[\frac{\partial(\ln n)}{\partial r} + (\frac{v^2}{V_T^2} - \frac{3}{2})\frac{\partial(\ln T)}{\partial r}\Big] f_M - v_\parallel \hat{b} \cdot \vec{\nabla} f_S \quad (3.50)$$

Let's try to solve this equation for the toroidal geometry we've been considering throughout these notes so far.

3.3.1 Spitzer Conductivity

$$C(f_S) = \frac{e}{k_BT} v_\parallel E_\parallel f_M \quad (3.51)$$

We have the Coulomb collision operator.

$$C_{coul}(f_e, f_i) = -\frac{\nu_{ei}}{2} v_{Te}^3 \frac{\partial}{\partial w_\alpha}\left[\left(\frac{\delta_{\alpha\beta}}{|w|} - \frac{w_\alpha w_\beta}{|w|^3}\right)\frac{\partial f_e}{\partial w_\beta}\right] \quad (3.52)$$

where $\vec{w} = \vec{v}_e - \vec{u}_i$. Let's try to get this into the form for trapped particles. (Question: how?) This form is

$$C(f) = 2\nu_{ei}(v) h \frac{\partial}{\partial \lambda}[\lambda \xi \frac{\partial}{\partial \lambda} f] \quad (3.53)$$

where $\xi = \frac{v_\parallel}{v} = \sqrt{1 - \frac{\lambda}{h}}$ where $\lambda = \frac{\mu B_0}{E}$. As usual, $h = 1 + \epsilon \cos\theta$ where $\epsilon = \frac{r}{R_0}$. Defining g_S to be $f_S = f_M g_S$, we have (using equation 3.51)

$$C(f_S) = 2\nu_{ei}(v) h \xi \frac{\partial}{\partial \lambda}[\lambda \xi \frac{\partial}{\partial \lambda} f_M g_s] = \frac{e}{k_BT} v_\parallel E_\parallel f_M$$

Since $C(f_M) = 0$, we can pull this out of both sides. We can also estimate the magnitude of the operators by assuming that $g_S \propto \xi$.[43] For trapped particles, $h \approx 1$, $\xi \approx \epsilon^{1/2}$, $\lambda \approx 1$, $\Delta\lambda \approx \epsilon^{1/2}$. This makes

$$h\frac{\partial}{\partial \lambda}[\lambda \xi \frac{\partial}{\partial \lambda} \xi] \approx \frac{h \lambda \xi^2}{\Delta \lambda^2} \approx 1$$

Actually, a more careful analysis suggests this is $\frac{1}{2}$ rather than 1. (why?) With this substitution, we get

$$\nu_{ei} f_M g_S = \frac{e}{k_BT} v_\parallel E_\parallel f_M$$

$$g_S = \frac{e}{k_BT\nu_{ei}} v_\parallel E_\parallel \quad (3.54)$$

$$f_S = g_S f_M = \frac{e}{k_BT\nu_{ei}} v_\parallel E_\parallel f_M \quad (3.55)$$

[43] The reason we make this assumption is justified in my GPP1 notes in the section on Lorentz conductivity. I don't want to explain it again.

The component of the current current driven by classical (Spitzer) resistivity is

$$J_{\|S} = -e\int v_\| f_S d^3\vec{v} = -e\int v_\| f_M g_S d^3\vec{v}$$

$$J_{\|S} = -\frac{e^2 E_\|}{k_B T}\int \frac{v_\|^2}{\nu_{ei}} f_M \tag{3.56}$$

To solve this integral, we use $\nu_{ei}(v) = \overline{\nu_{ei}}\frac{V_{Te}^3}{v^3}$, $v_\| = \xi v$, $d^3\vec{v} = 2\pi\xi d\lambda v^2 dv$ (why?). Eventually this becomes

$$J_{\|S} = \sigma_\| E_\| \tag{3.57}$$

$$\sigma_\| = \frac{32}{3\pi}\frac{n_0 e^2}{m_e \overline{\nu}_{ei}} \tag{3.58}$$

3.3.2 Back to calculating something

That was all calculating spitzer contribution to conductivity, based on f_S. But $f_e = f_0 + f_S + \hat{f}_e/$, and we need to calculate the contribution of $\hat{f}$ to the current, as given by

$$J_\| = -e\int v_\| f_e d^3\vec{v} = \sigma_\| E_\| - e\int v_\| \hat{f}_e d^3\vec{v} \tag{3.59}$$

This first term is of course the spitzer conductivity we just calculated, this second term is the neoclassical "bootstrap" current.

We also had the neoclassical electron flux, given by the first term in equation 2.40.

$$\Gamma_{Neo} \approx \frac{1}{eB_P}\langle R_{\|ei} - neE_\|\rangle \tag{3.60}$$

We're going to calculate this term as well. Remember, $\vec{R}_{ei} = \int m\vec{v}C_{ei}(f_e, f_i)d^3\vec{v}$. This means that

$$C_{ei}(f_e, f_i) = C_{ei}(\hat{f}, f_i) + C_{ei}(f_S, f_i) = C_{ei}(\hat{f}, f_i) + \frac{e}{k_B T}v_\| E_\| f_M$$

and, using $V_T^2 = \frac{2k_B T}{m}$ and $\int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \sqrt{\pi}/2$,

$$\frac{meE_\|}{k_B T}\int v_\|^2 f_M d^3\vec{v} = \frac{meE_\|}{k_B T}n\frac{v_T^2}{2} = enE_\|$$

$$R_{\|ei} = \int mv_\| C_{ei}(\hat{f}, f_i)d^3\vec{v} + \frac{meE_\|}{k_B T}\int v_\|^2 f_M d^3\vec{v} = \int mv_\| C_{ei}(\hat{f}, f_i)d^3\vec{v} + enE_\|$$

With these results, the neoclassical flux Γ_{Neo} simplifies.

$$\Gamma_{Neo} = \frac{1}{eB_P}\langle\int mv_\| C_{ei}(\hat{f}, f_i)d^3\vec{v} + enE_\| - neE_\|\rangle =$$

$$\frac{1}{eB_P}\langle\int mv_\parallel C_{ei}(\hat{f}, f_i)d^3\vec{v}\rangle \tag{3.61}$$

This implies that we can calculate the neoclassical particle flux if we can just calculate $\hat{f}$. Look back at equation 3.59: we can calculate the parallel current as well if we can calculate $\hat{f}$. The conclusion is that both quantities can be calculated if we just have $\hat{f}$. Let's set to calculate $\hat{f}$.

3.3.3 Calculating $\hat{f}$

Let's introduce the mathematically convenient form of the radial component of the radial drift velocity in a tokamak,

$$(v_D)_r = \frac{mv_\parallel}{qB_P}\hat{b}\cdot\vec{\nabla}(hv_\parallel) \tag{3.62}$$

How do we get it in this form? We start with our drift-velocity in the absence of electric fields,

$$\vec{v}_D = \frac{mv_\perp^2}{2qB^2}\hat{b}\times\vec{\nabla}B + \frac{mv_\parallel}{qB^2}\vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b} \tag{3.63}$$

In a low-β (i.e. a zero-$\vec{J}_\perp$ plasma), this becomes

$$\vec{v}_D = \frac{m(v_\perp^2/2 + v_\parallel^2)}{qB^2}\hat{b}\times\vec{\nabla}B \tag{3.64}$$

We prove this as follows: in a low-β plasma, we have the MHD equilibrium equation $\vec{\nabla}P = \vec{J}\times\vec{B}$ where $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}$. If the magnetic pressure is much higher than the plasma pressure, then $\vec{\nabla}P$ must be small which implies $\vec{J}_\perp$ must also be small. If $J_\perp = 0$, then $(\vec{\nabla}\times\vec{B})_\perp = 0$. We can use this to prove $\vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b} = \hat{b}\times\vec{\nabla}B$. We use a clever trick from Goldston's book.

$$0 = \hat{b}\times(\vec{\nabla}\times\vec{B})_\perp = \hat{b}\times(\vec{\nabla}\times\vec{B})$$
$$0 = \epsilon_{ijk}b_j(\epsilon_{klm}\partial_l B_m)$$
$$\epsilon_{kij}\epsilon_{klm}b_j\partial_l B_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})b_j\partial_l B_m = 0$$
$$b_j\partial_i B_j = b_j\partial_j B_i = \hat{b}\cdot(\vec{\nabla}\vec{B}) = (\hat{b}\cdot\vec{\nabla})\vec{B}$$
$$\hat{b}\times(\hat{b}\cdot(\vec{\nabla}\vec{B})) = \hat{b}\times(\hat{b}\cdot\vec{\nabla})\vec{B}$$

Now suppose at some point we choose our z-axis to be in the direction of the magnetic field. Then

$$\hat{b}\cdot(\vec{\nabla}\vec{B}) = \hat{z}\cdot(\frac{\partial B_z}{\partial x}\hat{x}\hat{z} + \frac{\partial B_z}{\partial y}\hat{y}\hat{z} + \frac{\partial B_z}{\partial z}\hat{z}\hat{z}) = \vec{\nabla}B_z = \vec{\nabla}B$$

and

$$\hat{b}\times(\hat{b}\cdot\vec{\nabla})\vec{B} = \hat{b}\times(\frac{\partial}{\partial z}(B_z\hat{b})) = \hat{b}\times\frac{\partial B_z}{\partial z}\hat{b} + \hat{b}\times(B_z\frac{\partial\hat{b}}{\partial z}) = \vec{B}\times\frac{\partial\hat{b}}{\partial z}$$

$$\hat{b} \times (\hat{b} \cdot \vec{\nabla})\vec{B} = \vec{B} \times (\hat{b} \cdot \vec{\nabla})\hat{b}$$

Putting this together, we have $\hat{b} \times \vec{\nabla} B = \vec{B} \times (\hat{b} \times \vec{\nabla})\hat{b}$. This proves that, for low-β plasmas, we can write our drift velocity as in equation 3.64.

Todo: how can we write this in terms of temperatures?

Now, using equation 3.64, we can write the drift velocity specifically in a tokamak. We have $|B| = \frac{B_0}{1+\epsilon \cos\theta} \approx B_0(1 - \epsilon \cos\theta)$, so that (now using cylindrical coordinates instead of toroidal coordinates) $\vec{\nabla} B = -\frac{B_0}{R_0}\hat{r}_{cylindrical}$ points radially inwards. This is illustrated in figure **??**. Since $\vec{B}$ is mostly in the toroidal direction, $\hat{b} \times \vec{\nabla} B = \frac{B_0}{R_0}\hat{z}$. The radial component of this is (now reverting back to toroidal coordinates) $\frac{B_0 \sin\theta}{R_0}$. With this result, the drift velocity in a tokamak becomes (using equation 3.64)

$$(v_D)_r = \frac{m(v_\perp^2/2 + v_\parallel^2)}{qBR_0}\sin\theta \tag{3.65}$$

Todo: Figure

Great. Let's try to get equation 3.62 in this form. For a collisionless particle in a tokamak,

$$v_\parallel = \sqrt{2/m}\sqrt{E - \mu B}$$

so

$$\vec{\nabla}(hv_\parallel) = v_\parallel \vec{\nabla} h + h\vec{\nabla} v_\parallel$$

$$\vec{\nabla}(hv_\parallel) = v_\parallel \frac{1}{R_0}\hat{r}_{cylindrical} - h\sqrt{\frac{1}{2m}}\frac{1}{\sqrt{E - \mu B}}\mu\vec{\nabla} B$$

$$\vec{\nabla}(hv_\parallel) = v_\parallel \frac{1}{R_0}\hat{r}_{cylindrical} - h\frac{1}{m}\frac{\mu\vec{\nabla} B}{v_\parallel}$$

Since $\vec{\nabla} B = -\frac{B_0}{R_0}\hat{r}_{cylindrical}$, then both of the above terms are in the (cylindrically) radially outwards direction. Since

$$\vec{B} = \frac{B_T}{h}\hat{\zeta} - \frac{B_P}{h}\hat{\theta}$$

(where the negative sign comes assuming J_ζ is in the positive direction) then (setting $h \approx 1$)

$$\hat{b} \cdot \vec{\nabla}(hv_\parallel) = \frac{B_P}{B_0}\frac{\sin\theta}{R_0}(v_\parallel + \frac{2}{m}\mu B_0)$$

$$\frac{mv_\parallel}{qB_P}\hat{b} \cdot \vec{\nabla}(hv_\parallel) = \frac{m\sin\theta}{qB_0R_0}(v_\parallel^2 + \frac{v_\perp^2}{2}) \tag{3.66}$$

Comparing equation 3.66 with equation 3.65, we see that these two expression are the same. This proves that equation 3.62 gives us our particles drifts in a tokamak.

Now let's try to solve for $\hat{f}$. Let's look back at equation 3.50, reproduced below.

$$v_\parallel \hat{b} \cdot \vec{\nabla} \hat{f} - C(\hat{f}) = -(v_D)_r \Big[\frac{\partial (\ln n)}{\partial r} + (\frac{v^2}{V_T^2} - \frac{3}{2}) \frac{\partial (\ln T)}{\partial r} \Big] f_M - v_\parallel \hat{b} \cdot \vec{\nabla} f_S \quad (3.67)$$

Let's remember how we got this equation. We started with the drift-kinetic equation for electrons in a toroidal geometry, and linearized f around a non-homogenous Maxwellian distribution function, where the first-order effects are due to drift velocity and electric fields. We then got an equation for the first-order distribution function f_1, which we separated into two parts: f_S, the first-order change in the distribution function due to the classical Spitzer resistivity (parallel electric field), and $\hat{f}_e$, the first-order change in the distribution function due to the toroidal effects. We're now solving for f_e, which we can do now that we know $(v_D)_r$. Plugging in our epression for $(v_D)_r$, we have

$$v_\parallel \hat{b} \cdot \vec{\nabla} \hat{f} - C(\hat{f}) = -\frac{m v_\parallel}{q B_P} \hat{b} \cdot \vec{\nabla}(h v_\parallel) \Big[\frac{\partial (\ln n)}{\partial r} + (\frac{v^2}{V_T^2} - \frac{3}{2}) \frac{\partial (\ln T)}{\partial r} \Big] f_M - v_\parallel \hat{b} \cdot \vec{\nabla} f_S$$

Fortunately, we've already calculated f_S, back in equation 3.55, reproduced below.

$$f_S = \frac{e}{k_B T \nu_{ei}} v_\parallel E_\parallel f_M$$

Taking the gradient gives

$$\vec{\nabla} f_S = \frac{e v_\parallel E_\parallel}{k_B T \nu_{ei}} \Big[\frac{\partial (\ln n)}{\partial r} + (\frac{v^2}{V_T^2} - \frac{3}{2}) \frac{\partial (\ln T)}{\partial r} \Big] f_M + \frac{e E_\parallel}{k_B T \nu_{ei}} \vec{\nabla} v_\parallel$$

we can also make the approximation $\hat{b} \cdot \vec{\nabla} v_\parallel \approx \hat{b} \cdot \vec{\nabla}(h v_\parallel)$ (why?). This gives us

Notice that we've got two terms which go like $v_\parallel \hat{b} \cdot \vec{\nabla}$

Notice that we've got two terms that a

4 Anomalous Transport

"For a successful technology, reality must take precedence over public relations, for nature cannot be fooled."

Richard Feynman

What is anomalous transport? Essentially, everything which enhances transport above neoclassical levels. That means waves and turbulence which might be generated in a toroidal geometry. Neoclassical transport happens even in steady-state, anomloulous transport requires some time-evolving plasma state to create that enhanced transport. In this class, we'll focus on two types of anomalous transport: drift waves, created by a density gradient in a plasma, and ion temperature gradient waves (ITG waves), which as their name suggests are created by temperature gradients in a plasma. The existence of a density gradient and temperature gradient are things which we expect to arise in a tokamak geometry, and indeed do.

Why are we studying drift waves and ITG waves? what does that have to do with anamolous transport?

Todo: make a table summarizing all of the variables, what they mean

4.1 Drift Waves

Let's start with the slab geometry shown in figure **??**. In this geometry, we have a magnetic field pointing in the z-direction, and a density gradient pointing in the x-direction. While this is a simple slab geometry, it isn't too unlike the geometry we would see in a tokamak. The toroidal direction in a tokamak is analogous to the z-direction in this slab geometry, as both have the magnetic field primarily in that direction. While in a tokamak the density gradient is mostly in the r-direction, here it is in the x-direction. So the x-direction is analogous to the r-direction in a tokamak. Similarly, the y-direction is analogous to the poloidal direction in a tokamak. To start to understand this, we'll look at the single-particle diamagnetic drift speed,

$$\vec{v}_{*\sigma} = \frac{T_\sigma}{q_\sigma B_0 n}\frac{dn}{dx}\hat{y} \tag{4.1}$$

Todo: write about how we got this

As we've mentioned, the density gradient sets up a current in our plasma. Let's suppose we end up finding some oscillation with wavenumber k_y. Then we can create some frequency out of the diamagnetic drift velocity and k_y.

$$\omega_{*\sigma} = k_y v_{*\sigma} = \frac{k_y k_B T_\sigma}{q_\sigma B_0}\frac{d(\ln n)}{dx} \tag{4.2}$$

Todo: figure

Todo: explain how diamagnetic current balances pressure gradient force so MHD equilibrium is stable

Now, we'll need to make some approximations to get a tractable answer here. Firstly, we need to assume that the Gyro radius $\rho_\sigma = \frac{k_B T_\sigma}{q_\sigma B_0}$ is small relative to the length scale of the density changes, $L = -\frac{1}{n}\frac{dn}{dx}$. We also assume that the frequency of the oscillation is small relative to the gyrofrequency, $\Omega = \frac{q_\sigma B}{m_\sigma}$. Actually, since the ion gyrofrequency is so much smaller than the electron gyrofrequency and the ion gyroradius is so much larger than the electron gyroradius, we need to worry about the ion quantities and not really the electron quantities. This can be written

$$\rho_i/L \ll 1$$

$$\omega/\Omega_i \ll 1$$

We also have that the diamagnetic drift velocity is much smaller than that species thermal velocity, by a factor ρ_σ/L.

$$v_{*\sigma} = \rho_\sigma \frac{V_{T\sigma}}{L}$$

We also have that

$$\frac{\omega}{\Omega} \sim \frac{\omega_*}{\Omega} \sim \frac{k_y \rho V_T}{\Omega L} \sim k_y V_T \frac{\rho}{L} \ll 1$$

Why do we care?

Since parallel to the magnetic field, the longest wavelength a wave can have is L_c, the connection length, then we expect $k_\parallel \geq \frac{1}{L_c}$. Drift waves operate in a parameter regime where

$$V_{Ti} < \frac{\omega}{k_\parallel} < V_{Te}$$

This means that, like the ion acoustic wave, drift waves operate in the parameter regime where electrons are isothermal and ions are adiabatic. In some sense, drift waves are like modified ion acoustic waves, except with a density gradient and a magnetic field. Now, drift waves are electrostatic in origin, meaning we'll solve them just like we solve any other electrostatic wave: using the momentum equation, continuity equation, and Poisson's equation. We linearize these equations, starting from the equilibrium we have in figure **??**. The equilibrium we treat using MHD for simplicity, which gives us

$$\vec{\nabla} P = \vec{J} \times \vec{B} \tag{4.3}$$

This tells us that the diamagnetic current balances the pressure gradient set up by the density gradient.

Now let's solve for the dispersion relation of drift waves in this geometry. Remember how we solved for the dispersion relation of the ion acoustic wave way back in GPP1? What we did was we found the adiabatic and isothermal response functions for each species, then plugged these response functions into Gauss's law. If you remember, the response function is the first-order density perturbation $n_{\sigma 1}$. In GPP1, we found the response functions using a kinetic approach and using a fluid approach, where we took a different limit of the solution to the equations depending on which limit we were in. We'll take that approach to finding the adiabatic ion response function. However, we take a different approach to finding the isothermal electron response function. Since the electrons are isothermal, they are approximately in thermal equilibrium. This means that they satisfy Boltzmann's equation

$$n_e = n_0 \exp\left[-\frac{E}{k_B T_e}\right] = n_0 \exp\frac{e\phi}{k_B T_e} \tag{4.4}$$

$$n_e \approx n_0\left[1 + \frac{e\phi_1}{k_B T_e}\right]$$

$$n_{e1} = n_0 \frac{e\phi_1}{k_B T_e} \tag{4.5}$$

Compare equation 4.5 with the isothermal response function we derived back in GPP1,

$$n_{\sigma 1} = -\frac{q_\sigma \phi}{m_\sigma V_{T\sigma}^2}$$

Setting $V_{T\sigma}^2 = \frac{k_B T_\sigma}{m_\sigma}$, we see that our two response functions are equal! We've actually stumbled upon a pretty cool trick for quickly getting isothermal response functions: just use Boltzmann's equation. It's a lot faster than solving the fluid or kinetic equations in an isothermal limit. Equation 4.5 gives us our electron response function. What about the ion response function? To solve for this, we're going to have to solve the fluid equations for the ions in this slab geometry. To solve the fluid equations, we linearize around our non-homogenous equilibrium, keeping all quantities to first-order. We'll assume that $\vec{B}$ has no first-order component, and $T_i = 0$. (why?) What do our linearized equations become? We have the linearized continuity equation and linearized momentum equation Todo: make all these $\vec{u}$ instead of $\vec{v}$.

$$\frac{\partial n_1}{\partial t} + \frac{\partial n_0}{\partial x} v_{1x} + n_0(\vec{\nabla} \cdot \vec{v}_1) = 0 \tag{4.6}$$

$$\frac{\partial \vec{v}_1}{\partial t} = \frac{e}{m_i}(-\vec{\nabla}\phi_1 + \vec{v}_1 \times \vec{B}_0) \tag{4.7}$$

where we've ignored the ion pressure term and any first-order change in $\vec{B}$. Next, we'll assume an exponential dependence in each of the first-order quantities. Todo: change all previous n's to n_0. This gives

$$\phi = \phi_1(x) \exp\left[i(k_y y + k_\parallel z - \omega t)\right]$$

$$n_i = n_0(x) + n_1 \exp\left[i(k_y y + k_\parallel z - \omega t)\right]$$

$$\vec{v}_i = \vec{v}_{i0} + \vec{v}_1 \exp\left[i(k_y y + k_\parallel z - \omega t)\right]$$

We want to solve the continuity equation to get an equation for the first-order density, so that we have our ion response function. To do so, we need equations for v_{1x} and $\vec{\nabla} \cdot \vec{v}_1$. We can solve the momentum equation to get $\vec{v}_1$ which allows us to solve the continuity equation. Solving the momentum equation by components, we have

$$-i\omega v_{1x} = \frac{e}{m_i}(v_{1y}B_0)$$

$$-i\omega v_{1y} = \frac{e}{m_i}(-ik_y\phi_1 - v_x B_0)$$

$$-i\omega v_{1z} = \frac{e}{m_i}(-ik_\parallel \phi_1)$$

Rearranging for the first-order velocities, we have

$$v_{1y} = -i\frac{\omega}{\Omega_i}v_{1x}$$

$$v_{1x} = i\frac{\omega}{\Omega_i}v_{1y} - i\frac{k_y}{B_0}\phi_1$$

$$v_{1z} = \frac{e}{m_i\omega}k_\parallel\phi_1 \tag{4.8}$$

Looking at the equations for v_{1x} and v_{1y}, we see that we have 2 equations but two unknowns. We can therefore solve these equations for v_{1x} and v_{1y}. Plugging the first equation into the second equation, we get

$$v_{1x} = \frac{\omega^2}{\Omega_i^2}v_{1x} - i\frac{k_y}{B_0}\phi_1$$

so

$$v_{1x} = \frac{1}{1-\omega^2/\Omega_i^2}\frac{-ik_y}{B_0}\phi_1 \tag{4.9}$$

which means

$$v_{1y} = \frac{1}{1-\omega^2/\Omega_i^2}\frac{-\omega e k_y}{m_i\Omega_i^2}\phi_1 \tag{4.10}$$

Remember what we're trying to do: solve for v_{1x} and $\vec{\nabla} \cdot \vec{v}$ so that we can solve the continuity equation for the ion response function. With equations 4.48, 4.49, and 4.8, we have $\vec{v}_1$. But since we're looking at drift waves where the frequency is much less than the ion cyclotron frequency, then we can use the approximation $\frac{\omega}{\Omega_i} \ll 1$ to write $\frac{1}{1-\omega^2/\Omega_i^2} \approx 1$. With this approximation, let's solve for $\vec{\nabla} \cdot \vec{v}_1$.

$$\vec{\nabla} \cdot \vec{v}_1 = -i\omega\Big[\frac{ek_y^2}{m_i\Omega_i^2} - \frac{ek_\parallel^2}{m_i\omega^2}\Big]\phi_1 \tag{4.11}$$

$$\vec{\nabla} \cdot \vec{v}_1 = -i\omega\frac{e\phi_1}{k_BT_e}\frac{k_BT_e}{m_i}\Big[\frac{k_y^2}{\Omega_i^2} - \frac{k_\parallel^2}{\omega^2}\Big]$$

$$\vec{\nabla} \cdot \vec{v_1} = -i\omega \frac{e\phi_1}{k_B T_e}\Big[b_s - \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big] \tag{4.12}$$

where $c_s^2 = \frac{2k_B T_e}{m_i}$, $\rho_s = \frac{c_s}{\Omega_i}$, and $b_s = \frac{1}{2}k_y^2\rho_s^2$. Remember that c_s is the sound speed or acoustic speed in a plasma. ρ_s is a new variable we haven't seen before, which I call the acoustic gyroradius. It represents the gyroradius a particle would have if it were traveling at the sound speed. b_s is a funny dimensionless variable which represents how short the wavelengths are in the perpendicular direction. If k_y is really small relative to the acoustic gyroradius, then we have big perpendicular wavelengths and b_s is big. Similarly, if k_y is really big, then we have short perpendicular wavelengths and b_s is also really large. With $\vec{\nabla} \cdot \vec{v}_1$ in hand, we can plug this into the linearized continuity equation to solve for the ion response function. Using equation 4.6, we have

$$-i\omega n_1 - i\frac{\partial n_0}{\partial x}\frac{ek_y}{m_i\Omega_i}\phi_1 - i\omega\frac{n_0 e\phi_1}{k_B T_e}\Big[b_s - \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big] = 0$$

$$n_1 = -\frac{1}{n_0}\frac{\partial n_0}{\partial x}\frac{n_0 e k_y \phi_1}{m_i \omega \Omega_i} - \frac{n_0 e\phi_1}{k_B T_e}\Big[b_s - \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big]$$

$$\frac{n_1}{n_0} = \frac{e\phi_1}{k_B T_e}\Big[\frac{\omega_*}{\omega} - b_s + \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big] \tag{4.13}$$

where

$$\omega_* \equiv -\frac{k_y k_B T_e}{m_i\Omega_i}\frac{\partial(\ln n_0)}{\partial x} = \frac{k_y T_e}{m_i\Omega_i}\frac{1}{L_n} \tag{4.14}$$

Todo: make sure the sign convention for L_n gets explained at some point.

This is the ion response function! With both the electron and ion response functions in hand, we can solve for the dispersion relation for drift waves by plugging them into Gauss's law.

$$-\vec{\nabla}^2\phi = \frac{1}{\epsilon_0}\sum q_\sigma n_\sigma$$

$$-(k_\parallel^2 + k_y^2)\phi_1 = \frac{e^2 n_0 \phi_1}{k_B T_e \epsilon_0}\Big[-1 + b_s - \frac{\omega_*}{\omega} - \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big]$$

$$\frac{\epsilon_0 k_B T_e}{e^2 n_0}(-k_\parallel^2 - k_y^2) = \Big[-1 - b_s + \frac{\omega_*}{\omega} + \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big]$$

Now, the LHS equals $-\frac{\lambda_{De}^2}{\lambda^2}$, which is much less than 1 as long as the wavelength of the oscillation is much longer than the Debye length, as we indeed expect it to be. We can therefore neglect this LHS term in favor of the 1 (and other terms) on the RHS. This gives us a new dispersion relation for drift waves.

$$1 + b_s - \frac{\omega_*}{\omega} - \frac{k_\parallel^2 c_s^2}{2\omega^2} = 0 \tag{4.15}$$

Suppose we take the long perpendicular wavelength, large-B limit which makes $b_s \ll 1$. This allows us to solve for ω.

$$\omega^2 - \omega\omega_* - \frac{1}{2}k_\parallel^2 c_s^2 = 0$$

If we take $\omega_* = 0$ (which is the limit where there is no density gradient), we recover the dispersion relation for ion acoustic waves

$$\omega = \frac{k_\parallel c_s}{\sqrt{2}}$$

If we have a density gradient, then we have

$$\omega = \frac{1}{2}\left[\omega_* \pm \sqrt{\omega_*^2 + 2k_\parallel^2 c_s^2}\right] \tag{4.16}$$

This gives us two solutions, one which is always positive and one which is always negative. A plot of the two solutions is shown in figure **??**. As we can see, as the wavelengths decrease ($k_\parallel$ increases) the drift waves become more and more like ion acoustic waves. The density gradient has the most significant impact on the large-λ waves.

Question: what do positive and negative signs mean? right-moving and left-moving?

Todo: summarize what we just did

Todo: look forward to the future - why we have investigated these waves, how we will look at kinetic destabilization of these waves.

4.1.1 Kinetic Destabilization of Electron Drift Waves

Let's think back to GPP1, when we derived the electron and ion response function in both the adiabatic and isothermal limits two separate ways. The first time round, we used the Vlasov equation to get an equation for $f_{\sigma 1}$, which we integrated to get $n_{\sigma 1}$. While this integral blew up in the denominator at $\frac{\omega}{k} = v_\parallel$, we we able to get a result by expanding the denominator in certain limits. The second time round, we used the fluid equations, and encountered no singularities as long as $\frac{\omega^2}{k^2} \neq \gamma V_{T\sigma}^2$. In other words, using the fluid equations didn't give us a problem except at a single point, but using kinetic equations our integral had a singularity which we didn't know how to resolve.

Question: why is it called electron drift waves and not just drift waves?

As we saw in section 4.1, drift waves are like ion acoustic waves in a plasma with a magnetic field and a density gradient. We solved for their dispersion relation using *fluid* equations, and found no singularities. It turns out that if we solve for the dispersion relation of a drift wave using a kinetic model, we get singularities, just like when we solved for the dispersion relation of ion acoustic waves using a kinetic model. Here, we'll investigate drift waves using a kinetic approach for the electrons and see that electron kinetic effects cause a destabilization of the mode. These kinetic effects can be thought of as wave-particle resonances which drive the instability.

We start with the electrostatic drift-kinetic equation for the electrons,

$$\frac{\partial f}{\partial t} + \vec{v}_{GC} \cdot \vec{\nabla} f - \frac{e}{m} E_\parallel \frac{\partial f}{\partial v_\parallel} = 0 \tag{4.17}$$

We then linearize the distribution function around a Maxwellian, so that $f = f_M + f_1$. Since we're solving for the electron response in the slab geometry so that we can understand wave-particle resonances and their effect on drift waves, we let the quantities n and T in the Maxwellian depend only on x, which makes sense since the zeroth-order density is a function of only x. Linearizing the drift-kinetic equation gives

$$\frac{\partial f_1}{\partial t} + \vec{v}_\parallel \hat{b} \cdot \vec{\nabla} f_1 + \vec{v}_D \cdot \vec{\nabla} f_M - \frac{e}{m} E_\parallel \frac{\partial f_M}{\partial v_\parallel} = 0 \tag{4.18}$$

Now, since

$$f_M = n_0(x) \Big(\frac{m}{2\pi k_B T(x)} \Big)^{3/2} \exp\Big(- \frac{mv^2}{2k_B T_e(x)} \Big) \tag{4.19}$$

then

$$\vec{\nabla} f_M = \hat{x} \Big[\frac{\frac{1}{n_0} \partial n_0}{\partial x} - \frac{3}{2T_e} \frac{\partial T_e}{\partial x} + \frac{mv^2}{2k_B T_e^2} \frac{\partial T_e}{\partial x} \Big] f_M$$

$$\vec{\nabla} f_M = \hat{x} \Big[\frac{\partial \ln n_0}{\partial x} + \frac{\partial \ln T_e}{\partial x} \Big(\frac{E}{k_B T} - \frac{3}{2} \Big) \Big] f_M$$

and our magnetic field is straight, so $\vec{v}_D = \frac{\vec{E} \times \vec{B}}{B^2} = \frac{-\vec{\nabla}\phi_1 \times \hat{b}}{B}$, which implies

$$\vec{v}_D \cdot \vec{\nabla} f_M = - \frac{1}{B} \frac{\partial \phi_1}{\partial y} \Big[\frac{\partial \ln n_0}{\partial x} + \frac{\partial \ln T_e}{\partial x} \Big(\frac{E}{k_B T} - \frac{3}{2} \Big) \Big] f_M \tag{4.20}$$

We also have that

$$- \frac{e}{m} E_\parallel \frac{\partial f_M}{\partial v_\parallel} = \frac{e}{k_B T_e} E_\parallel v_\parallel f_M$$

Putting all this together, we have

$$\frac{\partial f_1}{\partial t} + v_\parallel \hat{b} \cdot \vec{\nabla} f_1 - \frac{1}{B} \frac{\partial \phi_1}{\partial y} \Big[\frac{\partial \ln n_0}{\partial x} + \frac{\partial \ln T_e}{\partial x} \Big(\frac{E}{k_B T} - \frac{3}{2} \Big) \Big] f_M - \frac{e}{k_B T_e} \frac{\partial \phi_1}{\partial z} v_\parallel f_M = 0 \tag{4.21}$$

Now, we can solve this equation for f_1, by assuming that the perturbed quantities go as $e^{i(k_\parallel z + k_y y - \omega t)}$. This gives

$$(-i\omega + iv_\parallel k_\parallel) f_1 = ik_y \frac{\phi_1}{B} \Big[\frac{\partial \ln n_0}{\partial x} + \frac{\partial \ln T_e}{\partial x} \Big(\frac{E}{k_B T_e} - \frac{3}{2} \Big) \Big] f_M + iv_\parallel k_\parallel \frac{e\phi_1}{k_B T_e} f_M$$

Cancelling the i's, we get

$$(\omega - k_\parallel v_\parallel)f_1 = -\frac{e\phi_1}{k_B T_e} f_M \left[k_\parallel v_\parallel + \frac{k_y k_B T_e}{eB}\frac{\partial \ln n_0}{\partial x}\Big[1 + \frac{\partial \ln T_e}{\partial \ln n_0}\Big(\frac{E}{k_B T_e} - \frac{3}{2}\Big)\Big]\right]$$

We'll use some definitions to make this easier. We set

$$\omega_* \equiv -\frac{k_y k_B T_e}{eB}\frac{\partial(\ln n_0)}{\partial x} = \frac{k_y k_B T_e}{eB}\frac{1}{L_n}$$

$$\eta_e \equiv \frac{\partial \ln T_e}{\partial \ln n_0}$$

$$\omega_{T*} \equiv \omega_* \Big[1 + \eta_e\Big(\frac{E}{k_B T_e} - \frac{3}{2}\Big)\Big]$$

This gives us a nice expression for f_1.

$$f_1 = \frac{e\phi_1}{k_B T_e} f_M [\frac{\omega_{T*} - k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel}]$$

We can rewrite this in a more convenient form by adding and subtracting 1.

$$f_1 = \frac{e\phi_1}{k_B T_e} f_M [\frac{\omega_{T*} - k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel} - \frac{\omega - k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel} + 1]$$

$$f_1 = \frac{e\phi_1}{k_B T_e} f_M - \frac{e\phi_1}{k_B T_e}\Big[\frac{\omega - \omega_{T*}}{\omega - k_\parallel v_\parallel}\Big] f_M \tag{4.22}$$

To get the electron response function n_1, we need to integrate f_1 over velocities.

$$n_1 = \frac{e\phi_1}{k_B T_e}\left[\int f_M d^3\vec{v} - \int \Big(\frac{\omega - \omega_{T*}}{\omega - k_\parallel v_\parallel}\Big) f_M d^3\vec{v}\right] \tag{4.23}$$

The first term integrates to $n_0(x)$, which gives us the isothermal response function we derived for the electrons in section 4.1. The second term is where the fun happens, meaning where the kinetic effects come into play. To make things more complicared, ω_{T*} has a hidden factor of E in it, which means we'll have to really careful with the integration. Let's write it out carefully.

$$\int \Big(\frac{\omega - \omega_{T*}}{\omega - k_\parallel v_\parallel}\Big) f_M d^3\vec{v} = n_0 \int \Big(\frac{1 - \frac{\omega_{T*}}{\omega}}{1 - \frac{k_\parallel v_\parallel}{\omega}}\Big)\Big(\frac{m}{2\pi k_B T_e(x)}\Big)^{3/2} \exp\Big(-\frac{mv^2}{2k_B T_e(x)}\Big) d^3\vec{v}$$

When $\omega = k_\parallel v_\parallel$, this goes to infinity. So we should be able to integrate over $v_\perp$ without any problems. We've gotta be careful about the hidden factor of E, but in principle this can be done. Let's get to work on this second term.

$$= n_0 \int \Big(\frac{1}{1 - \frac{k_\parallel v_\parallel}{\omega}}\Big)\Big(\frac{m}{2\pi k_B T_e}\Big)^{1/2} \exp\Big(\frac{-mv_\parallel^2}{2k_B T_e(x)}\Big) dv_\parallel$$

$$\int\int\Big(1-\frac{\omega_*}{\omega}\Big[1+\eta\Big(\frac{v_\parallel^2+v_{\perp1}^2+v_{\perp2}^2}{2k_BT_e/m_e}-\frac{3}{2}\Big)\Big]\Big)\Big(\frac{m}{2\pi k_BT_e}\Big)\exp\Big(\frac{-m(v_{\perp1}^2+v_{\perp2}^2)}{2k_BT_e(x)}\Big)dv_{\perp1}dv_{\perp2}$$

This seems terrible - and it isn't a pretty expression by any means. But if we make it dimensionless, we see that the factor with η completely cancels when we do the Gaussian integrals. Setting $V_{Te}^2=\frac{2k_BT_e}{m_e}$, and $x=\frac{v_\parallel}{V_{Te}}$, $y=\frac{v_{\perp1}}{V_{Te}}$, and $z=\frac{v_{\perp2}}{V_{Te}}$, then $dv_\parallel=dx\sqrt{\frac{k_BT_e}{m_e}}$, etc. We can greatly simplify our integral then.

$$=n_0(x)\int\Big(\frac{1}{1-\frac{k_\parallel v_\parallel}{\omega}}\Big)\frac{1}{\sqrt{\pi}}\exp{(-x^2)}dx$$

$$\int\int\Big(1-\frac{\omega_*}{\omega}\Big[1+\eta\Big(x^2+y^2+z^2-\frac{3}{2}\Big)\Big]\frac{1}{\pi}\exp{(-y^2-z^2)}dydz\Big)$$

Let's do the integral with respect to z first. Since $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$ and $\int_{-\infty}^{\infty}x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$, then every term brings out a factor of $\sqrt{\pi}$ in front, except the term with z^2 which gets cut in half relative to the other terms. We are left with

$$=n_0\int\Big(\frac{1}{1-\frac{k_\parallel v_\parallel}{\omega}}\Big)\frac{1}{\sqrt{\pi}}\exp{(-x^2)}dx$$

$$\int\Big(1-\frac{\omega_*}{\omega}\Big[1+\eta\Big(x^2+y^2+\frac{1}{2}-\frac{3}{2}\Big)\Big]\frac{1}{\sqrt{\pi}}\exp{(-y^2)}dy\Big)$$

Essentially the same thing happens with the y-integral, giving us

$$=n_0\int\Big(\frac{1-\frac{\omega_*}{\omega}[1+\eta(x^2-\frac{1}{2})]}{1-\frac{k_\parallel v_\parallel}{\omega}}\Big)\frac{1}{\sqrt{\pi}}\exp{(-x^2)}dx$$

Actually, this integral isn't in the form we want it in yet, since $x=v_\parallel/V_{Te}$, yet we have a $v_\parallel$ in the bottom. We fix this by multiplying the numerator and denominator by $1/V_{Te}$ and pulling out a $k_\parallel/\omega$ from the bottom. This makes the second term we've been working on

$$=n_0\frac{\omega}{k_\parallel V_{Te}}\int\Big(\frac{1-\frac{\omega_*}{\omega}[1+\eta(x^2-\frac{1}{2})]}{\frac{\omega}{k_\parallel V_{Te}}-x}\Big)\frac{1}{\sqrt{\pi}}\exp{(-x^2)}dx$$

We're going to make an additional variable substitution, such that $\zeta_e\equiv\frac{\omega}{k_\parallel V_{Te}}$. We do this because we're going to introduce the plasma dispersion function, $Z(\zeta)$. The plasma dispersion function $Z(\zeta)$ is defined as

$$Z(\zeta)\equiv\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\exp{(-x^2)}}{x-\zeta}dx \tag{4.24}$$

As you can see, our second term is looking awfully like the plasma dispersion function, except one of the terms has an x^2 up top.

$$=-n_0\zeta_e\int\Big(\frac{1-\frac{\omega_*}{\omega}[1+\eta(x^2-\frac{1}{2})]}{x-\zeta_e}\Big)\frac{1}{\sqrt{\pi}}\exp{(-x^2)}dx$$

We are going to treat the x^2 term (how? why?). With this approximation, the integral over x^2 cancels the $\frac{1}{2}$, meaning we drop the η term and are left with

$$\int \Big(\frac{\omega - \omega_{T*}}{\omega - k_{\parallel} v_{\parallel}}\Big) f_M d^3\vec{v} = -n_0(1 - \frac{\omega_*}{\omega})\zeta_e Z(\zeta_e) \tag{4.25}$$

Remember, this is the second term in the electron response function. With this term solved for, we have the kinetic electron response function for the slab geometry in figure **??**. From equation 4.23, this is

$$n_1 = \frac{e\phi_1 n_0}{k_B T_e}\Big[1 + (1 - \frac{\omega_*}{\omega})\zeta_e Z(\zeta_e)\Big] \tag{4.26}$$

Todo: summarize what we've done so far in detail

4.1.2 Plasma Dispersion Function

Before we go any further, we're going to need to know some things about the plasma dispersion function $Z(\zeta_e)$. As we defined above, the plasma dispersion function is defined as

$$Z(\zeta) \equiv \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\exp{(-x^2)}}{x - \zeta} dx \tag{4.27}$$

Of course, this integral has a singularity at $x = \zeta_e$, which means we can't solve it unless we use complex analysis to evaluate it. We actually won't worry about solving it in the general case in these notes. Instead, we'll solve it in the $\zeta_e \ll 1$ limit.[44] In this limit, our function becomes

$$Z(\zeta) = i\pi^{1/2} \exp{(-\zeta^2)} - 2\zeta + \frac{4}{3}\zeta^2 - ... \tag{4.28}$$

We can actually show this, using a wickedly clever trick. Since (for $\mathrm{Im}(\zeta) > 0$, so that the integral converges)

$$i\int_0^{\infty} \exp{[-i(x-\zeta)\tau]} d\tau = i\Big[\frac{-\exp{[-i(x-\zeta)\tau]}}{i(x-\zeta)}\Big]_0^{\infty} = \frac{1}{x-\zeta}$$

then we can write $Z(\zeta)$ as

$$Z(\zeta) = \frac{i}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp{(-x^2)} dx \int_0^{\infty} \exp{[-i(x-\zeta)\tau]} d\tau \tag{4.29}$$

We can move terms around and complete the square to get it in the form we want.

$$Z(\zeta) = \frac{i}{\pi^{1/2}} \int_0^{\infty} \exp{(i\zeta\tau)} d\tau \int_{-\infty}^{\infty} \exp{(-x^2 - ix\tau)} dx$$

[44] Recognize that since $\zeta_e \equiv \frac{\omega}{k_{\parallel} V_{Te}}$, it makes sense we would want to evaluate $Z(\zeta_e)$ in the small-ζ_e limit. The reason for this is that drift waves, like ion acoustic waves, are in the regime $V_{Ti} \ll \frac{\omega}{k_{\parallel}} \ll V_{Te}$, since the electrons are isothermal but the ions are adiabatic.

$$Z(\zeta) = \frac{i}{\pi^{1/2}} \int_0^\infty \exp\left(i\zeta\tau - \frac{\tau^2}{4}\right)d\tau \int_{-\infty}^\infty \exp\Big(-x^2 - ix\tau + \frac{\tau^2}{4}\Big)dx$$

$$Z(\zeta) = \frac{i}{\pi^{1/2}} \int_0^\infty \exp\left(i\zeta\tau - \frac{\tau^2}{4}\right)d\tau \int_{-\infty}^\infty \exp\Big(-\Big(x + \frac{i\tau}{2}\Big)^2\Big)dx$$

This second integral, through a simple change of variables, becomes

$$\int_{y=-\infty+\frac{i\tau}{2}}^{y=\infty+\frac{i\tau}{2}} \exp(-y^2)dy = \sqrt{\pi}$$

The fact that the integral is the same as if there wasn't complex integers in the numerator has to do with the fact that the integral encloses no poles, and any closed integrals in the complex plane integrate to zero. Armed with those two pieces of knowledge, we can deform the integral along the real axis and convince ourselves it gives the same result as along the real axis but with a larger imaginary component. Having solved this integral, we have that

$$Z(\zeta) = i\int_0^\infty \exp\left(i\zeta\tau - \frac{\tau^2}{4}\right)d\tau \tag{4.30}$$

This is an alternative way of writing the plasma dispersion function, which will prove helpful to us as we continue in our studies of plasma physics. For now, though, let's just expand this integral in the small-ζ limit and see what we get.

$$Z(\zeta) = i\int_0^\infty e^{-\tau^2/4}\Big[1 + i\zeta\tau - \frac{\zeta^2\tau_2}{2} - i\frac{\zeta^3\tau^3}{3!} + \frac{\zeta^4\tau^4}{4!} + ...\Big]d\tau$$

$$Z(\zeta) = i\sqrt{\pi}(1 - \zeta^2 + ...) - 2\zeta\Big[1 - \frac{2\zeta^2}{3} + ...]\Big]$$

The first of these infinite summations ends up summing into something nice, $e^{-\zeta^2}$. The second term, as far as I know, doesn't give us anything nice. But what we're interested in (at least for the destabilization of drift waves) is the imaginary component of $Z(\zeta)$, since that gives us an exponentially growing or decaying component. So our imaginary component of $Z(\zeta)$ in the small-ζ limit is

$$Z_{Im}(\zeta) = i\sqrt{\pi}e^{-\zeta^2} \approx i\sqrt{\pi} \tag{4.31}$$

4.1.3 Returning to Drift Waves

We have what we need out of the plasma dispersion function, $Z(\zeta_e)$: the imaginary component of it in the $\zeta_e \ll 1$ limit. Now we can evaluate equation 4.26. We have

$$n_{e1} = \frac{e\phi_1}{k_B T_e} n_0 \Big[1 + i\sqrt{\pi}\frac{\omega - \omega_*}{k_\parallel V_{Te}}\Big] \tag{4.32}$$

We now have a nice expression for the kinetic electron response function in the slab geometry relevant to drift waves. The imaginary component is key, because it tells us that our waves will have some imaginary component, the sign of which depends on $\omega - \omega_*$ where ω_* is related to the strength of the density gradient. We would expect that if the density gradient is weak, we will see damping of the wave due to kinetic effects, but if the density gradient is sufficiently strong we'll see an exponential growth of the perturbation. Let's see how this plays out.

We have the ion response function which we calculated using the fluid method. We could calculate the kinetic response function of the ions, but what we have for the electrons is sufficient for our purposes. (why don't we calculate ion response function?) The ion response function is

$$n_{i1} = n_0 \frac{e\phi_1}{k_B T_e}\Big[\frac{\omega_*}{\omega} - b_s + \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big]$$

where

$$\omega_* \equiv -\frac{k_y k_B T_e}{m_i \Omega_i}\frac{\partial(\ln n_0)}{\partial x}$$

Plugging these into Gauss's law, and ignoring the $-k^2\phi_1$ term as before, we have

$$\frac{\omega_*}{\omega} - b_s + \frac{k_\parallel^2 c_s^2}{2\omega^2} - 1 = i\sqrt{\pi}\frac{\omega_* - \omega}{k_\parallel V_{Te}} \tag{4.33}$$

We can solve this equation perturbatively. To lowest order (assuming $k_\parallel$ and k_y are small, so we have large-λ waves and b_s and $\frac{k_\parallel^2 c_s^2}{\omega^2}$ are negligible) then we have, to lowest order, $\omega \approx \omega_*$. Now to the next-lowest order, we can write $\omega \approx \omega_* + \delta\omega$ where δ_ω has both real and imaginary components. Equation 4.33 becomes

$$\frac{\omega_*}{\omega_* + \delta\omega} - b_s + \frac{k_\parallel^2 c_s^2}{2(\omega_* + \delta\omega)^2} - 1 = -i\sqrt{\pi}\frac{\delta\omega}{k_\parallel V_{Te}}$$

$$1 - \frac{\delta\omega}{\omega_*} - 1 - b_s + \frac{k_\parallel^2 c_s^2}{2\omega_*^2}(1 - 2\frac{\delta\omega}{\omega_*}) = -i\sqrt{\pi}\frac{\delta\omega}{k_\parallel V_{Te}}$$

$$\frac{\delta\omega}{\omega_*} = \Big(\frac{k_\parallel^2 c_s^2}{2\omega_*^2} - b_s\Big)\Big/\Big(1 + \frac{k_\parallel^2 c_s^2}{2\omega_*^2} - \frac{i\sqrt{\pi}\omega_*}{k_\parallel V_{Te}})$$

Expanding the denominator, we have

$$\frac{\delta\omega}{\omega_*} = (\frac{k_\parallel^2 c_s^2}{2\omega_*^2} - b_s)(1 - \frac{k_\parallel^2 c_s^2}{2\omega_*^2} + \frac{i\sqrt{\pi}\omega_*}{k_\parallel V_{Te}}) \tag{4.34}$$

so

$$\frac{\delta\omega_r}{\omega_*} \approx (\frac{k_\parallel^2 c_s^2}{2\omega_*^2} - b_s) \tag{4.35}$$

$$\frac{\delta\omega_i}{\omega_*} \approx (\frac{k_\parallel^2 c_s^2}{2\omega_*^2} - b_s)\frac{\sqrt{\pi}\omega_*}{k_\parallel V_{Te}} \tag{4.36}$$

Now remember: our time-dependence was assumed to be $e^{-i\omega t}$. So a *positive* imaginary component corresponds to an exponentially growing solution. Todo

4.1.4 Electromagnetic Drift Waves

Question: what does this have to do with finite beta? Question: what does $A_\parallel$ and $A_\perp$ have to do with anything?

Todo: explain why we care

Let's look at figure **??** as we have throughout section 4.1, but not suppose we allow the possibility that our magnetic field is perturbed to first-order. This means, of course, that $\vec{B} = B_0\hat{z} + \vec{B}_1$. Since $\vec{B} = \vec{\nabla} \times \vec{A}$, then $\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$. Now, we choose to ignore compressional effects associated with $\vec{A}_\perp$, and instead focus on $A_\parallel$. (why? explain?) We also have

$$\vec{B}_1 = \vec{\nabla} \times A_z\hat{z} = \frac{\partial A_z}{\partial y}\hat{x} = ik_y A_z \hat{x}$$

where we've used the fact that first-order quantities go as $e^{ik_z z + ik_y y - i\omega t}$, and we've assumed that any change in x is small so that $\frac{\partial A_\parallel}{\partial x} \approx 0$. Since we're allowing for time-varying induced magnetic fields, then the $\frac{\partial \vec{A}}{\partial t}$ gives us induced electric fields as well.

$$\vec{E} = -\vec{\nabla}\phi_1 - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi_1 + i\omega A_z\hat{z} \tag{4.37}$$

The unit vector in the direction of the magnetic field, $\hat{b}$, now no longer points only in the z-direction.

$$\hat{b} = \frac{B_0}{\sqrt{B_0^2 + B_1^2}}\hat{z} + \frac{\vec{B}_1}{\sqrt{B_0^2 + B_1^2}} \approx \hat{z} + \frac{\vec{B}_1}{B_0}$$

Otherwise, we're looking at essentially the same drift wave. This means that, in order to solve for the dispersion relation of the wave, we're again going to find the electron and ion response functions, plug them into Gauss's law, and solve for ω as a function of k. The essential nature of the wave doesn't change: electromagnetic drift waves are still primarily electrostatic waves where the electrons are isothermal and the ions are adiabatic. Question: does using gauss's law to solve for wave dispersion relation imply wave is electrostatic? what does it imply? We're going to first solve for the response function of the electrons, using the drift-kinetic equation, and secondly solve for the response function of the ions. The drift-kinetic equation for electrons is

$$\frac{\partial f}{\partial t} + \vec{v}_{GC} \cdot \vec{\nabla} f - \frac{e}{m}E_\parallel \frac{\partial f}{\partial v_\parallel} = 0 \tag{4.38}$$

Now we linearize this equation, as we did in equation 4.18, except $\vec{v}_{GC}$ has a parallel component along the $\vec{B}_1$ direction, and not just the $\hat{z}$-direction. As before, we linearize around an Maxwellian distribution f_M, and solve for f_1, the first-order perturbation to f. We treat f_1, $\vec{B}_1$, and $E_\parallel$ as first order quantities. Linearizing and keeping only the terms to first order gives us

$$\frac{\partial f_1}{\partial t} + v_z \hat{z} \cdot \vec{\nabla} f_1 + v_z \frac{\vec{B}_1}{B_0} \cdot \vec{\nabla} f_M - \frac{\vec{\nabla}\phi_1 \times \hat{z}}{B_0} \cdot \vec{\nabla} f_M - \frac{e}{m} E_z \frac{\partial f_M}{\partial z} = 0 \tag{4.39}$$

The $\vec{E} \times \vec{B}$ drift term doesn't have the induced-$\vec{E}$ component because we have set $\vec{A} = A_z \hat{z}$, so crossing this with $\hat{z}$ gives us zero. However, the $E_\parallel$ term *does* include this term, so we need to consider it. From equation 4.37, we have

$$E_z = -ik_z\phi_1 + i\omega A_z \hat{z} = -ik_z(\phi_1 - \frac{A_z \omega}{k_z})$$

For simplicity, we're going to define

$$\psi \equiv \frac{A_z \omega}{k_z}$$

so that

$$E_z = -ik_z(\phi_1 - \psi)$$

Our usual expression for the gradient of a Maxwellian distribution in a slab geometry is

$$\vec{\nabla} f_M = \Big[\frac{\partial \ln n_0}{\partial x} + \frac{\partial \ln T_e}{\partial x}\Big(\frac{E}{k_B T_e} - \frac{3}{2}\Big)\Big] f_M \hat{x}$$

Here, however, we're going to assume that $\frac{\partial T_e}{\partial x} = 0$ for simplicity.[45] This means that

$$\vec{\nabla} f_M = \frac{\partial \ln n_0}{\partial x} f_M \hat{x}$$

We also need to calculate the derivative of a Maxwellian distribution with respect to v_z. Since a Maxwellian goes as $e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2k_B T_e}}$, this becomes

$$\frac{\partial f_M}{\partial v_z} = -\frac{m v_z}{k_B T_e} f_M$$

Lastly, we Fourier transform in space and time. After all these manipulations, our linearized drift-kinetic equation (equation 4.39) becomes

$$(-i\omega + ik_z v_z) f_1 = \Big(-ik_y v_z \frac{A_z}{B_0} + i\frac{k_y \phi_1}{B_0}\Big)\frac{\partial \ln n_0}{\partial x} f_M + \frac{e}{k_B T_e} ik_z v_z(\phi_1 - \psi) \tag{4.40}$$

Using $\psi = \frac{A_z \omega}{k_z}$, we can rewrite the first term inside the parentheses as

$$-i\frac{k_y}{B_0}\frac{k_z v_z}{\omega}\psi$$

[45] We're going to consider the effects of a temperature gradient later in these notes.

We can also use the definition

$$\omega_* = -\frac{k_y k_B T_e}{e B_0}\frac{\partial \ln n_0}{\partial x} \tag{4.41}$$

to rewrite the term with the parentheses as

$$-i\omega_* \frac{e}{k_B T_e}\Big(\phi_1 - \frac{k_z v_z}{\omega}\psi\Big) f_M$$

With these manipulations, our drift-kinetic equation becomes

$$(-i\omega + i k_z v_z) f_1 = -i\omega_* \frac{e}{k_B T_e}\Big(\phi_1 - \frac{k_z v_z}{\omega}\psi\Big) f_M + \frac{e}{k_B T_e} i k_z v_z (\phi_1 - \psi) f_M \tag{4.42}$$

Cancelling the i's, multiplying by -1, and dividing by $\omega - k_z v_z$, equation 4.40 becomes

$$f_1 = \frac{e}{k_B T_e}\Big[\frac{-k_z v_z(\phi_1 - \psi)}{\omega - k_z v_z} + \frac{\omega_*(\phi_1 - \frac{k_z v_z}{\omega}\psi)}{\omega - k_z v_z}\Big] f_M$$

$$f_1 = \frac{e}{k_B T_e}\Big[\Big(\frac{\omega_* - k_z v_z}{\omega - k_z v_z}\Big)\phi_1 + \Big(\frac{k_z v_z (1 - \frac{\omega_*}{\omega})}{\omega - k_z v_z}\Big)\psi\Big] f_M \tag{4.43}$$

This isn't a particularly nice expression, but we can simplify it somewhat if we expand in the limit $\frac{\omega}{k_z |v_z|} \ll 1$. Of course, v_z is the phase-space variable in f, so it can take on any possible value. *However*, when we integrate f_1 over velocity to get n_1, $|v_z|$ is typically of order V_{Te}. We also have that $\frac{\omega}{k_z} \sim v_{ph}$, the phase velocity of the wave. This means that our expansion is like taking $\frac{v_{ph}}{V_{Te}} \ll 1$, which makes sense because the phase velocity of drift waves is much slower than the thermal velocity of electrons. With this approximation, we can simplify equation 4.43.

$$f_1 = \frac{e}{k_B T_e}\Big[\Big(\frac{1 - \frac{\omega_*}{k_z v_z}}{1 - \frac{\omega}{k_z v_z}}\Big)\phi_1 - \Big(\frac{1 - \frac{\omega_*}{\omega}}{1 - \frac{\omega}{k_z v_z}}\Big)\psi\Big] f_M$$

$$f_1 \approx \frac{e}{k_B T_e}\Big[\phi_1 - (1 - \frac{\omega_*}{\omega})\psi + \frac{\omega}{k_z v_z}[(1 - \frac{\omega_*}{k_z v_z})\phi_1 - (1 - \frac{\omega_*}{\omega})\psi]\Big] f_M$$

$$f_1 \approx \frac{e}{k_B T_e}\Big[\phi_1 - (1 - \frac{\omega_*}{\omega})\psi\Big] f_M \tag{4.44}$$

If we integrate this over velocity space, we have the electron response function n_{1e}.

$$n_{1e} = n_0 \frac{e}{k_B T}\Big[\phi_1 - (1 - \frac{\omega_*}{\omega})\psi\Big] \tag{4.45}$$

Great, so we have the electron response function. Now we need to calculate the ion response function. As before, we'll use a fluid model to calculate the ion response function. Notice that this approach parallels the approached used in section 4.1.1, where we used a kinetic model for the electrons but a fluid model for the ions. The difference here is that our electric field is no longer purely electrostatic, but has an electromagnetic component as well. Once again, we assume the ions are cold, and ignore the effects of pressure. Our linearized fluid equations for the ions are

$$\frac{\partial n_1}{\partial t} + \frac{\partial n_0}{\partial x} v_{1x} + n_0(\vec{\nabla} \cdot \vec{u}_1) \tag{4.46}$$

$$\frac{\partial \vec{u}_1}{\partial t} = \frac{e}{m}(-\vec{\nabla}\phi_1 - \frac{\partial \vec{A}_1}{\partial t} + \vec{u}_1 \times \vec{B}_0) \tag{4.47}$$

Compare equations 4.46 and 4.47 with equations 4.6 and 4.7. As you can see, the only difference is with the electric field term in the momentum equation, where we now have an added $-\frac{\partial \vec{A}}{\partial t}$. Notice that the first-order change in the magnetic field, $\vec{B}_1$, doesn't show up in the momentum equation because we don't have a zeroth-order velocity term. Now, we can solve these equations as we did before, first taking the Fourier transform and then solving for $\vec{u}_1$ using the linearized momentum equation and plugging that into the linearized continuity equation. By components, the linearized momentum equation is

$$-i\omega u_{1x} = \frac{e}{m_i}(u_{1y} B_0)$$

$$-i\omega u_{1y} = \frac{e}{m_i}(-ik_y \phi_1 - u_x B_0)$$

$$-i\omega u_{1z} = \frac{e}{m_i}(-ik_z \phi_1 - i\omega A_z)$$

The x and y equations are the same as in the original drift wave case, but the z-equation has a term with A_z. This means we can rearrange for the x and y velocities exactly as we did before. Only the z-velocity is different than before. This gives us to get

$$u_{1x} = \frac{1}{1 - \omega^2/\Omega_i^2} \frac{-ik_y}{B_0} \phi_1 \tag{4.48}$$

$$u_{1y} = \frac{1}{1 - \omega^2/\Omega_i^2} \frac{-\omega e k_y}{m_i \Omega_i^2} \phi_1 \tag{4.49}$$

$$u_{1z} = \frac{e}{m_i}(\frac{k_z}{\omega}\phi_1 + A_z) \tag{4.50}$$

As before, since $\omega/\Omega_i \ll 1$, we can replace the $1 - \omega^2/\Omega_i^2$ with 1. We can now plug these velocities into the linearized continuity equation. However, this requires solving for $\vec{\nabla} \cdot \vec{u}_1$ first.

$$\vec{\nabla} \cdot \vec{u}_1 = -i\omega(\frac{e k_y^2 \phi_1}{m_i \Omega_i^2} - \frac{e k_z^2}{m_i}(\frac{\phi_1}{\omega^2} + \frac{A_z}{k_z \omega}))$$

$$\vec{\nabla} \cdot \vec{u}_1 = -i\omega \frac{e}{k_B T_e} \frac{k_B T_e}{m_i} (\frac{k_y^2}{\Omega_i^2}\phi_1 - \frac{k_z^2}{\omega^2}(\phi_1 - \psi))$$

$$\vec{\nabla} \cdot \vec{u}_1 = -i\omega \frac{e}{k_B T_e}\Big[b_s \phi_1 - \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)\Big]$$

where $c_s^2 = \frac{2k_B T_e}{m_i}$, $\rho_s = \frac{c_s}{\Omega_i}$, and $b_s = \frac{1}{2}k_y^2 \rho_s^2$. Notice that this is the same as equation 4.12, except we have an additional ψ term, and of course we've replaced $k_\parallel$ with k_z because the z-direction is no longer necessarily parallel to the magnetic field.

Having solved for $\vec{\nabla} \cdot \vec{u}_1$, we can plug the velocities into our linearized continuity equation, equation 4.46.

$$-i\omega n_1 - i\frac{1}{n_0}\frac{\partial n_0}{\partial x}\frac{e k_y n_0 \phi_1}{m_i \Omega_i} - i\omega n_0 \frac{e}{k_B T_e}\Big[b_s \phi_1 - \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)\Big] = 0$$

$$\frac{n_1}{n_0} = \frac{e}{k_B T_e}[\frac{\omega_*}{\omega}\phi_1 - b_s \phi_1 + \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)]$$

where we've used

$$\omega_* = -\frac{k_y k_B T_e}{e B_0}\frac{\partial \ln n_0}{\partial x}$$

Rearranging slightly, we have our ion response function for electromagnetic drift waves.

$$\Big(\frac{n_1}{n_0}\Big)_i = \frac{e}{k_B T_e}\Big[(\frac{\omega_*}{\omega} - b_s)\phi_1 + \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)\Big] \tag{4.51}$$

Great, so we have our electron and ion response functions. Todo: summarize what we've done, and what we're going to. Next, we plug our electron and ion response functions into Gauss's law, and solve for the dispersion relation.

$$\vec{\nabla}\phi_1 = -\sum_\sigma q_\sigma n_{\sigma 1}$$

$$k_y^2 \phi_1 + k_z^2 \phi_1 = n_0 \frac{e^2}{k_B T_e}\Big[\phi_1 - (1 - \frac{\omega_*}{\omega})\psi\Big] - n_0 \frac{e^2}{k_B T_e}\Big[(\frac{\omega_*}{\omega} - b_s)\phi_1 + \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)\Big]$$

$$(k_y^2 + k_z^2)\frac{k_B T_e}{n_0 e^2}\phi_1 = k^2 \lambda_{De}^2 \phi_1 = \Big[\phi_1 - (1 - \frac{\omega_*}{\omega})\psi\Big] - \Big[(\frac{\omega_*}{\omega} - b_s)\phi_1 + \frac{k_z^2 c_s^2}{2\omega^2}(\phi_1 - \psi)\Big]$$

Since $k^2 \lambda_{De}^2 = \frac{\lambda_{De}^2}{\lambda^2}$, and for drift waves we expect to see large-wavelength modes, then the LHS of Gauss's law is much less than 1, and we can neglect it relative to the RHS.[46] The RHS of the above expression gives us our dispersion relation.

$$(1 - \frac{\omega_*}{\omega} + b_s - \frac{k_z^2 c_s^2}{2\omega^2})\phi_1 + (-1 + \frac{\omega_*}{\omega} + \frac{k_z^2 c_s^2}{2\omega^2})\psi = 0 \tag{4.52}$$

This is great, and we'd be able to solve for $\omega(k)$, except we have a problem: we don't know what ψ is. Of course, it's defined as $\psi = \frac{A_z \omega}{k_z}$, but we still don't

[46] This is the same approximation we saw in an earlier section, I just wanted to explicitly write it out again so we remembered how it was done.

know anything about A_z. So let's try to solve for A_z. Todo: explain what we're gonna do. To start, we have Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Using $\vec{B} = \vec{\nabla} \times \vec{A}$, and assuming our time-dependence is slow so that we can ignore the displacement current, this becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla}^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \mu_0 \vec{J}$$

If we work in Coulomb gauge, then this becomes

$$-\vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

The z-component of this equation is

$$(k_y^2 + k_z^2) A_z = \mu_0 J_z$$

which gives

$$A_z = \frac{\mu_0 J_z}{k_y^2 + k_z^2}$$
$$\psi = \frac{\mu_0 J_z \omega}{k_z (k_y^2 + k_z^2)} \tag{4.53}$$

Great, so we have an expression for ψ! So we can plug this into equation 4.51, and solve for the dispersion relation, right? Well, yes and no. We could, but we still don't have an expression for J_z. You might be frustrated with how many steps it is taking to solve this stupid dispersion relation. I certainly am. But I promise, this is the last thing we'll need to do to get our dispersion relation. We can see the light at the end of the tunnel. To reach that light, we again use Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We're neglecting the displacement current, because the phase velocity of the time-change of E is slow relative to the speed of light. This gives us

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

which we can take the divergence of to get

$$\vec{\nabla} \cdot \vec{J} = 0$$
$$i k_z J_z + \vec{\nabla} \cdot \vec{J}_\perp = 0$$
$$J_z = \frac{i}{k_z} \vec{\nabla} \cdot \vec{J}_\perp \tag{4.54}$$

So to solve for J_z, we need to solve for $\vec{\nabla} \cdot \vec{J_\perp}$. However, we can solve for $\vec{J_\perp}$, using

$$\vec{J_\perp} = en_0(\vec{u}_{\perp i} - \vec{u}_{\perp e}) \tag{4.55}$$

Fortunately, we already know what $\vec{u}_{\perp i}$ and $\vec{u}_{\perp e}$ are. For the isothermal electrons, the perpendicular velocity is just the drift velocity,

$$\vec{u}_{\perp e} = -\vec{\nabla}\phi \times \hat{b}/B_0 = -ik_y\phi_1/B_0\hat{x}$$

For the adiabatic ions, we've calculated their velocities using the fluid equations already. Using u_{1x} and u_{1y}, we have

$$\vec{u}_{\perp i} = \frac{-ik_y\phi_1}{B_0}\hat{x} - \frac{\omega e k_y \phi_1}{m_i\Omega_i^2}\hat{y}$$

As you can see, the $\hat{x}$ terms from the ions and the electrons are the same, so these cancel when we calculate the perpendicular current. This leaves us only with a y-component of the perpendicular current, which gives us

$$\vec{J_\perp} = -n_0\phi_1(\frac{\omega e^2 k_y}{m_i\Omega_i^2})\hat{y}$$

$$\vec{\nabla} \cdot \vec{J_\perp} = -i\omega\frac{n_0 e^2 k_y^2}{m_i\Omega_i^2}\phi_1$$

Using equation 4.54, we have for J_z

$$J_z = \frac{\omega n_0 e^2 k_y^2}{k_z m_i \Omega_i^2}\phi_1$$

Using $\Omega_i = \frac{eB_0}{m_i}$, this becomes

$$J_z = \frac{\omega n_0 m_i k_y^2}{k_z B_0^2}\phi_1 \tag{4.56}$$

Great! We've reached the light at the end of the tunnel. Now we can plug our result into equation 4.53, and then plug *that* result into equation 4.52 to get an equation for ω. Plugging our result into equation 4.53, we have

$$\psi = \frac{\mu_0\omega}{k_z(k_y^2 + k_z^2)}\frac{\omega n_0 m_i k_y^2}{k_z B_0^2}\phi_1 = \frac{\mu_0 n_0 m_i}{B_0^2}\frac{\omega^2}{k_z^2}\phi_1 = \frac{\omega^2}{v_A^2 k_z^2}\phi_1 \tag{4.57}$$

where we've used the approximation that $k_y^2 + k_z^2 \approx k_y^2$, which is true because (why? todo: fix this). Plugging this result into equation 4.52, we have

$$(1 - \frac{\omega_*}{\omega} + b_s - \frac{k_z^2 c_s^2}{2\omega^2})\phi_1 + (-1 + \frac{\omega_*}{\omega} + \frac{k_z^2 c_s^2}{2\omega^2})\frac{\omega^2}{v_A^2 k_z^2}\phi_1 = 0$$

The factors of 1 and $\frac{\omega_*}{\omega}$ cancel, and we are left with

$$b_s = (1 - \frac{\omega_*}{\omega} - \frac{k_z^2 c_s^2}{2\omega^2})(1 - \frac{\omega^2}{v_A^2 k_z^2})$$

Todo: solve this dispersion relation
Todo: recap everything that we just did
Todo: explain why it matters

4.2 Nonlocal Analysis

In section 4.1, we looked at a slab geometry where we had a magnetic field in the z-direction, and a density gradient in the x-direction. In this geometry, our equilibrium was accomplished because the pressure gradient due to the density gardient was balanced by the diamagnetic current, crossed with the magnetic field. In this geometry, the z-direction represented the toroidal direction, the x-direction represented the radial direction, and the y-direction represented the poloidal direction. Crucially, this geometry had symmetry in the y/poloidal direction, and no magnetic field in this direction. This simplified our analysis greatly. However, it's not so realistic when we're considering a real-life tokamak, for a number of reasons. One of the reasons this geometry isn't realistic is that there *isn't* a poloidal magnetic field. In this section, we look at the effect of adding a poloidal magnetic field to the slab geometry.

Todo: figure 4nl

If we want to investigate the effects of a poloidal magnetic field, we're going to have to give the magnetic field a component in the y-direction. If the toroidal current is being carried diffusely across the poloidal cross-section of the tokamak, then we expect the poloidal magnetic field to grow like $\sim r$.[47] This means that in our slab geometry, we should let the y-component of the magnetic field be proportional to x. This means that

$$\vec{B} = B_0(\hat{z} + \frac{x}{L_s}\hat{y}) \tag{4.58}$$

where L_s is some length scale. So our slab geometry has a more realistic magnetic field. It also has a density gradient, again in the x-direction as before. I've illustrated this geometry in figure **??**. Since we're using this geometry to investigate a tokamak, we can expect that $\frac{x}{L_s} \ll 1$, so the poloidal magnetic field is much smaller than the toroidal magnetic field. Of course, we still haven't solved for L_s. The trick to doing so is to realize that when $x = L_s$, then we have $B_T = B_P$. What this tells us is that physically, L_s is the *radial* distance at which the poloidal field would be the same as the toroidal field. Bill calls it the "magnetic shear length", because it tells us the lengthscale at which the purely toroidal magnetic field at the center of the poloidal cross-section of the

[47]We can convince ourselves of this with a simple Ampere's law calculation, letting $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$.

tokamak has become sheared at 45°. For an illustration of this effect, see figure ??.

Todo: figure 4tokamakShear

We can calculate L_s by remembering that $q = \frac{d\zeta}{d\theta}$. If we follow a field line in a tokamak a distance ds, then we have $d\zeta \approx \frac{B_T}{B}\frac{ds}{R}$, and $d\theta \approx \frac{B_P}{B}\frac{ds}{r}$, so

$$q \approx \frac{d\zeta}{d\theta} = \frac{B_T r}{B_P R} \tag{4.59}$$

For our slab geometry here, $B_T = B_0$, and at $r = L_s$ then $B_P = B_0$. This allows us to solve for L_s, using the above expression.

$$q \approx \frac{L_s}{R}$$

$$L_s \approx qR \approx qR_0$$

question: why does this in the notes have a q' and a r in it?

Great, so we have our new geometry we're going to be investigating. Remember that in this new geometry, z corresponds to the toroidal direction, x to the radial direction, and y to the poloidal direction. Since our magnetic field line is no longer pointing along $\hat{z}$, then we have

$$\hat{b} = \frac{B_z}{\sqrt{B_z^2 + B_y^2}}\hat{z} + \frac{B_y}{\sqrt{B_z^2 + B_y^2}}\hat{y}$$

This means that a derivative parallel to the field line becomes

$$\hat{b} \cdot \vec{\nabla} = \frac{B_z}{\sqrt{B_z^2 + B_y^2}}\frac{\partial}{\partial z} + \frac{B_y}{\sqrt{B_z^2 + B_y^2}}\frac{\partial}{\partial y}$$

Since $\hat{b}$ points perpendicular to the field line, and $\vec{\nabla} \to i\vec{k}$ when we Fourier transform, then we have that $\hat{b} \cdot \vec{\nabla} = k_\parallel$.

However, in tokamak coordinates, this looks a bit different, mainly because we can assume that the toroidal magnetic field is significantly stronger than the poloidal magnetic field. This means that $\hat{b} \approx \hat{\zeta} + \frac{B_P}{B_T}\hat{\theta}$ The gradient in toroidal coordinates is

Todo: write gradient in toroidal coordinates equation

which means that

$$\hat{b} \cdot \vec{\nabla} = \frac{B_P}{B_T}\frac{1}{rh}\frac{\partial}{\partial \theta} + \frac{1}{R_0 h}\frac{\partial}{\partial \zeta}$$

where (using equation 4.59) $\frac{B_P}{B_T} \approx \frac{r}{qR_0}$.[48] This gives

[48]Note that this make sense intuitively. q is a measure of more of less how many times a field line goes around toroidally before it goes around poloidally. So if q is large, we would expect that the poloidal magnetic field is not very strong compared to the toroidal magnetic field, which this expression suggests.

$$\hat{b} \cdot \vec{\nabla} = \frac{1}{qR}\frac{\partial}{\partial \theta} + \frac{1}{R_0 h}\frac{\partial}{\partial \zeta} \tag{4.60}$$

Now suppose we have some drift wave in a tokamak which is an (n, m) mode. Mathematically, this means that the perturbation of some quantity ϕ (here I will let ϕ be the electric potential and ϕ_1 be the perturbation to ϕ) goes as

$$\phi_1 = \mathrm{Re}(\phi_1(r)e^{in\zeta + im\theta - i\omega t})$$

Physically, a (n, m) mode means that the perturbation of whatever quantity we're interested in (here that quantity is electric potential ϕ, but it could be density n, temperature T, displacement ξ, etc) has n complete cycles in the toroidal direction and m complete cycles in the poloidal direction. An $n = 0$ mode means that there is no toroidal variation in the perturbation of that quantity, while an $n = 1$ mode means we have one maximum and one minimum as we go around toroidally. Similarly, an $m = 0$ mode means that there is no poloidal variation of the perturbation. An $m = 1$ mode means that there is one poloidal maximum and one poloidal minimum as we go around poloidally.[49] Make sure you can visualize what $(n = 1, m = 1)$ and $(n = 1, m = 2)$ modes look like.

For a drift wave which is in an (n, m) mode, we have (using equation 4.60)

$$|k_\parallel| \approx |\frac{1}{\phi_1}\hat{b} \cdot \vec{\nabla}\phi_1|$$

$$k_\parallel \approx \frac{n}{R} + \frac{m}{qR} \tag{4.61}$$

Todo: justify intuitively why this is true

Question: where does minus sign come from?

This is a nice result, because it tells us that if $m = -qn$, then we have $k_\parallel = 0$. Actually, this is a pretty intuitive result. Todo: draw figure where $q = 2$, $m = -2$, and $n = 1$. Todo: explain the figure. Todo: what is a rational surface?

Suppose our drift wave is propogating at $r = r_0$ along a rational (n, m) surface. While $k_\parallel = 0$ at this rational surface at r_0, it is true that q changes slightly as a function of r. This means that if we want to calculate $k_\parallel$ near $r = r_0$, we can Taylor expand equation 4.61 around $r = r_0$. This gives us

$$k_\parallel(r) \approx \frac{-m}{Rq^2}\frac{dq}{dr}(r - r_0)$$

Since $m = -qn$ on the rational surface, this becomes

$$k_\parallel \approx \frac{n}{Rq}\frac{dq}{dr}(r - r_0)$$

[49] Actually, all of this is a bit complicated than I'm making it because we have oscillation of the poloidal direction at the same time as the toroidal direction, so it doesn't really make sense to talk about a poloidal or toroidal maximum in the first place because for a fixed ζ, we have variation in θ, and for fixed θ we have variation in ζ. But I think you get the idea.

We also have that (using the gradient in toroidal coordinates)

$$k_\theta = |\frac{1}{\phi_1}\frac{1}{rh}\frac{\partial}{\partial\theta}\phi_1| = \frac{m}{rh}$$

Plugging this into the expression for $k_\parallel$, we have

$$k_\parallel = -k_\theta \frac{rq'}{q}\frac{1}{Rq}(r - r_0) = -\frac{k_\theta rq'}{q}\frac{1}{L_s}(r - r_0) \tag{4.62}$$

We can applying this expression to the slab geometry. Before we do so, we'll have to solve for q in the slab geometry.

$$q \approx \frac{B_T r}{B_p R} \approx \frac{B_0 x}{B_0 \frac{x}{L_s} R_0 h} = \frac{L_s}{R}$$

$$\frac{dq}{dr} \approx \frac{d}{dr}(\frac{L_s}{R_0}(1 - \frac{r\cos\theta}{R_0})) \approx -\frac{L_s}{R_0^2}$$

With q and q' solved for, we have (using 4.62)

$$k_\parallel = k_y x \frac{L_s}{R_0^2}\frac{R}{L_s}\frac{1}{L_s}(x)$$

Todo: understand everything that just happened

When we looked at drift waves in a slab geometry without a "poloidal" magnetic field in the y-direction, we found the electron and ion responses, plugged those into Gauss's law, and solved for the dispersion relation. We found the electron response first using the Boltzmann equation for a system in equilibrium, and later using the drift-kinetic equation to determine the kinetic destabilization of drift waves due to the density gradient. We calculated the ion response function using the fluid equations, assuming $T_i = 0$. We linearized the fluid equations, assumed an exponential dependence $e^{ik_z z + ik_y y - i\omega t}$, and solved for n_{1i}.

Now that we have a poloidal magnetic field, we are going to take a similar approach. We're going to use a kinetic approach to solving the electron response function, and a zero-temperature fluid approach to solving the ion response function. Now, it turns out that the electron response function is the same as without the additional magnetic field.

$$n_{1e} = n_0 \frac{e\phi_1}{k_B T_e}(1 - i\delta) \tag{4.63}$$

where $\delta = \sqrt{\pi}\frac{\omega - \omega_*}{k_\parallel V_{Te}}$ Question: how do we know the electron response function will be the same? It's the ion response function which gets modified. Let's solve for the ion response function, so that we can plug that into Gauss's law and solve for the dispersion relation. As usual, we have the linearized continuity equation and linearized momentum equation for the ions.

$$\frac{\partial n_1}{\partial t} + \frac{\partial n_0}{\partial x}u_{1x} + n_0(\vec{\nabla}\cdot\vec{u}_1) = 0 \tag{4.64}$$

$$\frac{\partial \vec{u}_1}{\partial t} = \frac{e}{m_i}(-\vec{\nabla}\phi_1 + \vec{u}_1 \times \vec{B}) \tag{4.65}$$

The difference between these equations and ones we've looked at previously is that $\vec{B}$ has a zeroth-order poloidal magnetic field. However, we're going to look at a different exponential dependence. Instead of allowing for k_y and k_z, we're going to take the limit $k_z \to 0$. Question: physically what does this limit mean? In this limit, our exponential dependence becomes $e^{ik_y y - i\omega t}$. With this exponential dependence, we are ready to solve for the ion response function. Using the continuity equation, we have

$$-i\omega n_1 = -\frac{\partial n_0}{\partial x}u_{1x} - n_0(\vec{\nabla}\cdot\vec{u}_1) = n_0\frac{1}{L_n}u_{1x} - n_0(\vec{\nabla}\cdot\vec{u}_1)$$

where we've used our usual definition for L_n, $L_n = -\frac{1}{n_0}\frac{\partial n_0}{\partial x}$. Solving this gives

$$\frac{n_1}{n_0} = \frac{i}{\omega}\Big[\frac{u_{1x}}{L_n} - (\vec{\nabla}\cdot\vec{u}_1)\Big] \tag{4.66}$$

Great, so (as usual) if we can solve for $\vec{u}_1$ then we have our ion response function. As usual, we'll do this using the continuity equation, writing the continuity equation by components, taking the Fourier transform and solving for each component of velocity. Remember, our magnetic field is

$$\vec{B} = B_0\hat{z} + B_0\frac{x}{L_s}\hat{y}$$

so by components, the continuity equation is

$$-i\omega u_{1x} = \frac{e}{m_i}(-\frac{\partial \phi_1}{\partial x} + u_{1y}B_0 - u_{1z}\frac{x}{L_s}B_0)$$

$$-i\omega u_{1y} = \frac{e}{m_i}(-ik_y\phi_1 - u_{1x}B_0)$$

$$-i\omega u_{1z} = \frac{e}{m_i}(u_{1x}\frac{x}{L_s}B_0)$$

We can solve these equations with some nifty algebra. Here we go![50]

$$u_{1x} = i\frac{eB_0}{\omega m_i}(-\frac{1}{B_0}\frac{\partial \phi_1}{\partial x} + u_{1y} - u_{1z}\frac{x}{L_s}) \tag{4.67}$$

$$u_{1y} = \frac{eB_0}{\omega m_i}(\frac{k_y}{B_0}\phi_1 - iu_{1x}) \tag{4.68}$$

$$u_{1z} = i\frac{eB_0}{\omega m_i}\frac{x}{L_s}u_{1x} \tag{4.69}$$

Plugging the expressions for u_{1y} and u_{1z} into u_{1x}, we have

[50] A la Mario.

$$u_{1x} = i\frac{\Omega_i^2}{\omega^2}(-\frac{m_i\omega}{eB_0^2}\frac{\partial \phi_1}{\partial x} + \frac{k_y}{B_0}\phi_1 - iu_{1x} - i\frac{x^2}{L_s^2}u_{1x})$$

$$u_{1x}(1 - \frac{\Omega_i^2}{\omega^2}(1 + \frac{x^2}{L_s^2})) = -i\frac{\Omega_i}{\omega}\frac{1}{B_0}\frac{\partial \phi_1}{\partial x} + i\frac{\Omega_i^2}{\omega^2}\frac{k_y}{B_0}\phi_1$$

$$u_{1x}\frac{\Omega_i^2}{\omega^2}(1 - \frac{\omega^2}{\Omega_i^2} + \frac{x^2}{L_s^2}) = i\frac{\Omega_i}{\omega}\frac{1}{B_0}\frac{\partial \phi_1}{\partial x} - i\frac{\Omega_i^2}{\omega^2}\frac{k_y}{B_0}\phi_1$$

$$u_{1x}(1 - \frac{\omega^2}{\Omega_i^2} + \frac{x^2}{L_s^2}) = i\frac{\omega e}{m_i\Omega_i^2}\frac{\partial \phi_1}{\partial x} - i\frac{ek_y}{m_i\Omega_i}\phi_1 \tag{4.70}$$

Now, the terms in parentheses on the LHS is approximately just 1. This is because we're looking in the limit $x/L_s \ll 1$, and since we're investigating drift waves we have $\frac{\omega}{\Omega_i} \ll 1$. When we're solving for u_{1x}, to first-order we can just ignore the term in parentheses, and treat it as 1. *However*, it turns out that when we solve for u_{1y}, the terms to zeroth and first order in the parameters $\frac{\omega}{\Omega_i}$ and $\frac{x}{L_s}$ will go to zero, and we're going to have to worry about the second-order terms. It turns out that $\frac{\partial \phi_1}{\partial x}$ is *also* small, because the variation in x is slow, so we treat it as first-order as well. So keeping terms to second order, we have

$$u_{1x} \approx \Big[i\frac{\omega e}{m_i\Omega_i^2}\frac{\partial \phi_1}{\partial x} - i\frac{ek_y}{m_i\Omega_i}\phi_1(1 + \frac{\omega^2}{\Omega_i^2} - \frac{x^2}{L_s^2})\Big]$$

Great, so we have the x-component of the first-order velocity. Now we need to do some more algebra to get u_{1y} and u_{1z}. Fortunately, we already have expressions for these variables in terms of u_{1x}. These are equations 4.68 and 4.69. This gives us, keeping terms to second order

$$u_{1y} = \frac{\Omega_i}{\omega}\Big[\frac{ek_y}{m_i\Omega_i}\phi_1 + \Big(\frac{\omega e}{m_i\Omega_i^2}\frac{\partial \phi_1}{\partial x} - \frac{ek_y}{m_i\Omega_i}\phi_1(1 + \frac{\omega^2}{\Omega_i^2} - \frac{x^2}{L_s^2})\Big)\Big]$$

$$u_{1y} = \frac{e}{m_i\Omega_i}\frac{\partial \phi_1}{\partial x} - \frac{ek_y\omega}{m_i\Omega_i^2}\phi_1 + \frac{ek_y}{m_i\omega}\frac{x^2}{L_s^2}\phi_1 \tag{4.71}$$

$$u_{1z} = \frac{\Omega_i}{\omega}\frac{x}{L_s}\Big[-\frac{\omega e}{m_i\Omega_i^2}\frac{\partial \phi_1}{\partial x} + \frac{ek_y}{m_i\Omega_i}\phi_1\Big] \tag{4.72}$$

Great, so equations 4.70,[51] 4.71, and 4.72 give us $\vec{u}_1$, which we can use to plug into equation 4.66 to solve for the ion response function. First, though, we need to work out $\vec{\nabla} \cdot \vec{u}_1$. Since we've taken the limit $k_z \to 0$, then $\frac{\partial}{\partial z} = 0$. This means that we only have to worry about the x and y-components of $\vec{u}_1$.

$$\vec{\nabla} \cdot \vec{u}_1 = i\frac{\omega e}{m_i\Omega_i^2}\frac{\partial^2 \phi_1}{\partial x^2} - i\frac{ek_y}{m_i\Omega_i}\frac{\partial \phi_1}{\partial x} + i\frac{ek_y}{m_i\Omega_i}\frac{\partial \phi_1}{\partial x} - i\frac{ek_y^2\omega}{m_i\Omega_i^2}\phi_1 + i\frac{ek_y^2}{m_i\omega}\frac{x^2}{L_s^2}\phi_1$$

[51] This equation has the term in parentheses on the LHS set to 1, because the two terms are small as we discussed before. The only reason we didn't set this to 1 outright was because we needed the second-order terms to get u_{1y} to work out.

The second and third terms cancel to give us

$$\vec{\nabla} \cdot \vec{u}_1 = i\frac{\omega e}{m_i \Omega_i^2}\frac{\partial^2 \phi_1}{\partial x^2} + i\frac{e k_y^2}{m_i \omega}(\frac{x^2}{L_s^2} - \frac{\omega^2}{\Omega_i^2})\phi_1 \tag{4.73}$$

We plug this equation into equation 4.66, to get

$$\Big(\frac{n_1}{n_0}\Big)_i = \Big[\frac{e k_y}{\omega m_i \Omega_i L_n}\phi_1 - \frac{e}{m_i \Omega_i^2 L_n}\frac{\partial \phi_1}{\partial x} + \frac{e}{m_i \Omega_i^2}\frac{\partial^2 \phi_1}{\partial x^2} + \frac{e k_y^2}{m_i \omega^2}(\frac{x^2}{L_s^2} - \frac{\omega^2}{\Omega_i^2})\phi_1\Big]$$

$$\Big(\frac{n_1}{n_0}\Big)_i = -\frac{1}{L_n}\frac{e}{m_i \Omega_i}\Big[\frac{1}{\Omega_i}\frac{\partial \phi_1}{\partial x} - \frac{k_y}{\omega}\phi_1\Big] + \frac{e}{m_i \Omega_i^2}\Big[\frac{\partial^2}{\partial x^2} - k_y^2\Big]\phi_1 + \frac{e}{m_i}\frac{k_y^2}{\omega^2}\frac{x^2}{L_s^2}\phi_1$$

We can simplify our response function further, using our definitions $\omega_* = \frac{k_y k_B T_e}{e B_0 L_n}$, $c_s^2 = \frac{2 k_b T_e}{m_i}$, and $\rho_s = \frac{c_s}{\Omega_i}$. It turns out that we can also neglect the term $-\frac{1}{L_n}\frac{\partial \phi_1}{\partial x}$ relative to the term $\frac{\partial^2 \phi_1}{\partial x^2}$, because Question: huh? Whats with the note? This gives us

$$\Big(\frac{n_1}{n_0}\Big)_i = \frac{e}{k_B T_e}\left[\frac{k_B T_e k_y}{\omega L_N e B_0}\phi_1 + \frac{k_B T_e}{m_i \Omega_i^2}\Big[\frac{\partial^2}{\partial x^2} - k_y^2\Big]\phi_1 + \frac{k_B T_e}{m_i}\frac{k_y^2 x^2}{\omega^2 L_s^2}\right]$$

$$\Big(\frac{n_1}{n_0}\Big)_i = \frac{e}{k_B T_e}\left[\frac{\omega_*}{\omega} + \frac{\rho_s^2}{2}\Big[\frac{\partial^2}{\partial x^2} - k_y^2\Big] + \frac{c_s^2 k_y^2 x^2}{2\omega^2 L_s^2}\right]\phi_1 \tag{4.74}$$

We have our non-local ion response function. Before we go any further, let's step back for a second and make sure we understand what we've done so far. Todo: summarize. Now, notice what the difference between the radially local and radially non-local response functions is. The radially local response function is (equation 4.13)

$$\frac{n_1}{n_0} = \frac{e}{k_B T_e}\Big[\frac{\omega_*}{\omega} - b_s + \frac{k_\parallel^2 c_s^2}{2\omega^2}\Big]\phi_1$$

This is the same as the radially non-local ion response function (equation 4.74), except we've taken $b_s \phi_1 \to \frac{\rho_s^2}{2}(k_y^2 - \frac{\partial^2}{\partial x^2})\phi_1$ and $k_\parallel \to \frac{k_y x}{L_s}$. Now that we have the ion and electron response functions, we can plug them into Gauss's law to get the dispersion relation. Remember, the electron response function is given by equation 4.63. As usual, we apply the quasineutrality condition for drift waves, so we ignore the $-k^2\phi_1$ term to get (cancelling the factor of $\frac{e^2 n_0}{k_B T_e}$)

$$\Big[\frac{\omega_*}{\omega} - b_s + \frac{\rho_s^2}{2}\Big[\frac{\partial^2}{\partial x^2} - k_y^2\Big] + \frac{c_s^2 k_y^2 x^2}{2\omega^2 L_s^2} - 1 + i\delta\Big]\phi_1 = 0 \tag{4.75}$$

Great, so we just have to solve this equation to get our dispersion relation. On the face of it, this looks like a complicated equation we might not be able to solve. However, it turns out that this equation can be written as the Weber equation

$$[A\frac{\partial^2}{\partial x^2} + B - Cx^2]\phi = 0 \tag{4.76}$$

where A, B, and C are constants equal to

$$A = \frac{\rho_s^2}{2}$$

$$B = -1 + i\delta - \frac{\rho_s^2 k_y^2}{2} + \frac{\omega_*}{\omega}$$

$$C = -\frac{c_s^2 k_y^2 x^2}{2\omega^2 L_s^2}$$

It turns out the solutions to this equation are the Hermite functions.[52] This means that can write the solution for ϕ_1 as

$$\phi_1 = \sum_l a_l H_l(\sigma^{\frac{1}{2}} x) \exp\left(-\sigma x^2/2\right) \tag{4.77}$$

It turns out that

$$\sigma = \pm\sqrt{C/2A}$$

$$A\sigma(2l+1) = B$$

Let's show that this is true. First, we need to know a couple properties of Hermite polynomials. We have that

$$H_n'(x) = 2nH_{n-1}(x)$$

We also have that

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x)$$

which means

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x)$$

Let's look at one particular l for ϕ_1, and find $\frac{\partial^2 \phi_1}{\partial x^2}$ for that ϕ_1. With this result, we'll use the Weber equation (equation 4.76) to match the coefficients A, B, and $C4$ for each l. This will help us todo. We can rewrite our solution for ϕ_1 in terms of the variable $y = \sigma^{\frac{1}{2}} x$.

$$\phi_1 = \sum_l a_l H_l(y) \exp\left(-y^2\right)$$

This also means that $\frac{\partial}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \sigma^{\frac{1}{2}}\frac{\partial}{\partial y}$. Solving for a single-$l$ component of ϕ_1, we have

$$\frac{\partial \phi_1}{\partial y} = -y\phi_1 + H_l'(y)e^{-y^2}$$

[52]These are the same Hermite functions as are used for the solution of the quantum harmonic oscillator. This makes sense, because the Weber equation is in the same form as the Schrodinger equation for the quantum harmonic oscillator. So we expect the form of the solutions to be the same.

$$\frac{\partial^2\phi_1}{\partial y^2} = -\phi_1 + y^2\phi - y\phi_1' - yH_l'(y)e^{-y^2} + H_l''(y)e^{-y^2}$$

$$\frac{\partial^2\phi_1}{\partial y^2} = -\phi_1 + 2y^2\phi_1 - 2yH_l'(y)e^{-y^2} + \frac{\partial}{\partial y}(2lH_{n-1}(y))e^{-y^2}$$

$$\frac{\partial^2\phi_1}{\partial y^2} = -\phi_1 + 2y^2\phi_1 - 4ylH_{l-1}(y)e^{-y^2} + (4ylH_{n-1}(y) - 2lH_n(y))e^{-y^2}$$

$$\frac{\partial^2\phi_1}{\partial y^2} = -\phi_1 + 2y^2\phi_1 - 2l\phi_1$$

$$\frac{\partial^2\phi_1}{\partial y^2} = 2y^2\phi_1 - (2l+1)\phi_1$$

$$\frac{\partial^2\phi_1}{\partial x^2} = 2\sigma^2x^2\phi_1 - \sigma(2l+1)\phi_1 \tag{4.78}$$

Note that this was solved for a *single-l* mode. Plugging this result into the Weber equation, we have (again for a single-l)

$$(2\sigma^2A - C)x^2\phi_1 + (B - \sigma(2l+1)A)\phi_1$$

This tells us that

$$\sigma = \pm\sqrt{\frac{C}{2A}} \tag{4.79}$$

$$B = \sigma(2l+1)A \tag{4.80}$$

Question: factor of 2

Now, there are two possible signs for σ. It turns out that the correct choice of sign depends on the boundary conditions on the Weber equation. Since for $x \to \infty$, $\phi_1 \to 0$, then we need to choose the negative value for σ. Question: but it's imaginary, so why do we need to choose negative to prevent exponentially growing/shrinking?

Question: WKB stuff

Question: all of last page? huh?

4.3 Ion Temperature Gradient Mode

Todo: summarize physics

To solve for the ion temperature gradient dispersion relation, we're going to follow the approach we've been using throughout this chapter to solve for the dispersion relation of various drift waves. We'll solve for the ion and electron response functions, plug those into Gauss's law, and use the fact that a drift wave is a slow wave to ignore the $-k^2\phi_1$ term in Gauss's law. What makes this wave different from the other drift waves we've been looking at is that now there is a temperature gradient in x in addition to the density gradient in x. While before we've set $\eta = \frac{\partial \ln T_{i0}}{\partial \ln n_0}$ to zero, now we're going to keep it and consider the effects of having a finite temperature gradient. The slab geometry we're considering is shown in figure **??** We're not going to consider the kinetic effects of the isothermal electrons, and simply use Boltzmann's equation to solve for the electron response function. To get the ion response function, we're going to use a fluid model, allowing for the possibility that there is a first-order pressure perturbation P_1. Let's start with the electrons.

Todo: make figure 4itgSlab

Since drift waves are in the frequency range

$$V_{Ti} \ll \frac{\omega}{k_\parallel} \ll V_{Te}$$

then the thermal velocity of electrons is much faster than the phase velocity of the wave. This means that electrons are *isothermal*, and we can treat them as a population in thermal equilibrium. For a system in thermal equilibrium, the states are populated with the Boltzmann distribution. The Boltzmann distribution tells us that the probability a particle is in a state s is related to the energy of the state s, and proportional to an exponential factor e^{-E/k_BT}. For a plasma in equilibrium, the probability of a particle of charge q being at a certain point in space as opposed to any other point in space is proportional to $e^{-q\phi/k_BT}$, where $q\phi$ is the electric potential energy of the particle. If we call n_0 the density of particles where $\phi = 0$, then we have (for electrons, where $q = -e$)

$$n_e = n_0 e^{\frac{e\phi}{k_BT_e}}$$

This is just the Boltzmann distribution, applied to a plasma in equilbrium. For plasmas where the thermal energy is much greater than the electric potential energy,[53] then we can expand the exponential in the small parameter $\frac{e\phi}{k_BT_e}$ to get

$$n_e = n_0(1 + \frac{e\phi}{k_BT_e})$$

This tells us that the first-order density perturbation for isothermal electrons is, ignoring kinetic effects,

$$n_{1e} = n_0 \frac{e\phi_1}{k_BT_e} \tag{4.81}$$

This is the electron response function. It's the same electron response function we used in the original drift wave calculation, where we treated the electrons as isothermal and ignored kinetic effects. Now let's calculate the ion response function, using a fluid model. Since we're working with the slab geometry in figure **??**, we have a straight magnetic field in the z-direction and a zeroth-order pressure $P_0(x)$. We'll assume that our first-order quantities go like $e^{ik_\parallel z + ik_y y - i\omega t}$, which allows us to replace any derivatives with respect to t, z, or y with the respective variable. As usual, we're going to use the continuity equation to get an equation for n_1 in terms of the first-order velocity $\vec{u}_1$, and the momentum equation to solve for the first-order velocity $\vec{u}_1$. In the momentum equation, we're actually going to ignore any first-order variations in pressure perpendicular to the magnetic field. (Question: why?) This is equivalent to

[53] Question: does this have anything to do with strongly coupled plasmas?

assuming that in the direction perpendicular to the magnetic field, the only velocity particles have is their guiding center drift velocity. We'll see that if we ignore any variation in the first-order pressure perpendicular to the magnetic field (mathematically, this means $\vec{\nabla} P_1 = ik_\parallel P_1 \hat{z}$), using the fluid model is equivalent to using a single-particle model for the perpendicular velocities. Either of these approaches will allow us to solve for u_{1x} and u_{1y}. However, to solve for $u_\parallel$, we'll need to allow for the possibility of a first-order pressure variation along the magnetic field. After we Fourier transform the parallel component of the momentum equation, we'll have an equation for $u_\parallel$ in terms of P_1. To solve for P_1, we'll need to use the linearized, Fourier transformed energy equation for adiabatic processes. Once we have the components of $\vec{u}_1$ solved for, we can plug them into the continuity equation to solve for the ion response function. Our linearized continuity equation for the ions is

$$\frac{\partial n_1}{\partial t} + \frac{\partial n_0}{\partial x} u_{1x} + n_0 (\vec{\nabla} \cdot \vec{u}_1) = 0 \tag{4.82}$$

Taking the Fourier transform, this becomes

$$-i\omega n_1 = -n_0 u_{1x} \frac{\partial \ln n_0}{\partial x} - n_0 (\vec{\nabla} \cdot \vec{u}_1)$$

$$\left(\frac{n_1}{n_0}\right)_i = \frac{i}{\omega} \left[\frac{u_{1x}}{L_n} - \vec{\nabla} \cdot \vec{u}_1\right] \tag{4.83}$$

Great, so as promised we've used the continuity equation to solve for n_1 as a function of $\vec{u}_1$. Now we need to use the momentum equation to solve for $\vec{u}_1$. The linearized momentum equation for the ions is

$$m_i \frac{\partial \vec{u}_1}{\partial t} = e(-\vec{\nabla}\phi_1 + \vec{u}_1 \times \vec{B}) - \frac{1}{n_0} \vec{\nabla} P_1 \tag{4.84}$$

Remember, we're assuming that $\vec{\nabla} P_1 = ik_\parallel P_1 \hat{z}$ (no perpendicular variation in the first-order density perturbation) because (???). The perpendicular components of this equation are, after Fourier transforming,

$$-i\omega u_{1x} = \frac{eB}{m_i} u_{1y}$$

$$-i\omega u_{1y} = -\frac{e}{m_i} i k_y \phi_1 - \frac{eB}{m_i} u_{1x}$$

Solving for u_{1x} and u_{1y},

$$u_{1x} = i\frac{\Omega_i}{\omega} u_{1y}$$

$$u_{1y} = \frac{\Omega_i}{\omega} \frac{k_y \phi_1}{B} - i\frac{\Omega_i}{\omega} u_{1x}$$

$$u_{1y} = \frac{\Omega_i}{\omega} \frac{k_y \phi_1}{B} + \frac{\Omega_i^2}{\omega^2} u_{1y}$$

$$u_{1y}(1 - \frac{\Omega_i^2}{\omega^2}) = \frac{\Omega_i}{\omega}\frac{k_y\phi_1}{B}$$
$$u_{1y}(1 - \frac{\omega^2}{\Omega_i^2}) = -\frac{\omega}{\Omega_i}\frac{k_y\phi_1}{B}$$

Since the frequency of drift waves is so much slower than the ion cyclotron frequency, the term on parenthese on the LHS is just equal to 1, and we have

$$u_{1y} = -\frac{\omega}{\Omega_i}\frac{k_y\phi_1}{B} \tag{4.85}$$

Plugging this in to our expression for u_{1x}, we have

$$u_{1x} = -ik_y\frac{\phi_1}{B} \tag{4.86}$$

Great, so we have our perpendicular velocities u_{1y} and u_{1x}. Notice, however, that these velocities are identical to the guiding-center drift velocities due to the $\vec{E} \times \vec{B}$ drift and the polarization drift. The $\vec{E} \times \vec{B}$ drift is

$$\vec{v}_E = \frac{-\vec{\nabla}\phi_1 \times \vec{B}}{B^2} = -ik_y\phi_1\frac{1}{B}\hat{x}$$

which is exactly u_{1x}, given by equation 4.86. From GPP1, you may remember the polarization drift as

$$\vec{v}_p = \frac{1}{\Omega B}\frac{d\vec{E}_\perp}{dt}$$

In our slab geometry, the polarization drift for the ions is

$$\vec{v}_p = \frac{-i\omega}{\Omega_i B}\vec{E}_\perp = -\frac{\omega k_y\phi_1}{\Omega_i B}\hat{y}$$

which is exactly u_{1y}, given by equation 4.85. Note that there is no curvature of the magnetic field, so we wouldn't expect to see any $\vec{\nabla}B$ or curvature drift terms. Indeed, we don't. We only see drifts due to $\vec{E}$. The conclusion is that if we ignore variations in the first-order pressure perturbation perpendicular to the magnetic field, the perpendicular fluid velocity $\vec{u}_1$ is the same as the single-particle drifts $\vec{v}_p$ and $\vec{v}_E$. Nevertheless, we've solved for the perpendicular velocities. However, we'll need to use the parallel component of the momentum equation to solve for $u_\parallel$. It is in this parallel component of the momentum equation where we'll have to worry about the effects of having finite ion temperature. The parallel component is

$$m_i\frac{\partial u_\parallel}{\partial t} = -e\vec{\nabla}_\parallel\phi_1 - \frac{1}{n_0}\vec{\nabla}P_1$$

Taking the Fourier transform of this equation, we have

$$-i\omega m_i u_\parallel = -ik_\parallel e\phi_1 - \frac{ik_\parallel}{n_0}P_1$$

Cancelling the factor of $-i$ and solving for $u_\parallel$, we have

$$u_\parallel = \frac{k_\parallel}{\omega m_i}(e\phi_1 + \frac{P_1}{n_0})$$

This can be rewritten in terms of c_s^2, and $P_0 = n_0 k_B T_i$.

$$u_\parallel = \frac{k_\parallel}{\omega}\frac{k_B T_e}{m_i}\Big[\frac{e}{k_B T_e}\phi_1 + \frac{T_i}{T_e}\frac{P_1}{P_0}\Big]$$

$$u_\parallel = \frac{k_\parallel c_s^2}{2\omega}\Big[\frac{e\phi_1}{k_B T_e} + \frac{T_i}{T_e}\frac{P_1}{P_0}\Big] \tag{4.87}$$

We've solved for the parallel first-order velocity perturbation due to a density and temperature gradient using the momentum equation. However, we've solved for it in terms of the first-order pressure variation, P_1, which we *don't* know yet. However, we can use the energy equation (also called the equation of state) to solve for P_1.

$$\frac{d}{dt}\Big(\frac{P}{n^\gamma}\Big) = 0 \tag{4.88}$$

As we've done with the continuity equation and the momentum equation, we linearize the energy equation and Fourier transform to give us an equation for the first-order variable, in this case P_1. Setting $P = P_0 + P_1$, $n = n_0 + n_1$, and remembering that $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$, this becomes

$$\frac{d}{dt}\Big(\frac{P_1}{n_0^\gamma} + \frac{P_0}{n_0^\gamma}(1 - \gamma\frac{n_1}{n_0})\Big) = 0$$

Since P_0 and n_0 do not change with time, and the fluid perpendicular velocity $\vec{u}_\perp$ is a first-order quantity (since it's proportional to ϕ_1) this becomes

$$\frac{1}{n_0^\gamma}\frac{\partial P_1}{\partial t} + \vec{u}\cdot\vec{\nabla}\Big(\frac{P_0}{n_0^\gamma}\Big) - \gamma\frac{P_0}{n_0^\gamma}\frac{1}{n_0}\frac{\partial n_1}{\partial t} = 0$$

$$\frac{1}{n_0^\gamma}\frac{\partial P_1}{\partial t} + \frac{1}{n_0^\gamma}\vec{u}\cdot\vec{\nabla}P_0 - \gamma\frac{P_0}{n_0^\gamma}\frac{1}{n_0}\vec{u}\cdot\vec{\nabla}n_0 - \gamma\frac{P_0}{n_0^\gamma}\frac{1}{n_0}\frac{\partial n_1}{\partial t} = 0$$

Multiplying by n_0^γ, and using the linearized continuity equation

$$-\frac{1}{n_0}\frac{\partial n_1}{\partial t} - \frac{1}{n_0}\vec{u}_1\cdot\vec{\nabla}n_0 = \vec{\nabla}\cdot\vec{u}_1$$

our energy equation becomes

$$\frac{\partial P_1}{\partial t} + \vec{u}\cdot\vec{\nabla}P_0 + \gamma P_0\vec{\nabla}\cdot\vec{u}_1 = 0$$

Relative to the parallel velocity, the perpendicular fluid velocity (which is just the guiding-center drift velocity of individual particles as we've established) is small, so we can say that $\vec{\nabla}\cdot\vec{u}_1 \approx \vec{\nabla}_\parallel u_\parallel$. (is this right?) We also have that

$\vec{\nabla} P_0$ points in the x-direction, since both the density and temperature gradients are in the x-direction. This means that $\vec{u}_1 \cdot \vec{\nabla} P_0 = u_{1x} \frac{\partial P_0}{\partial x}$, where u_{1x} is the $\vec{E} \times \vec{B}$ drift of the guiding center of particles. Our linearized energy equation is therefore

$$\frac{\partial P_1}{\partial t} + u_{1x} \frac{\partial P_0}{\partial x} + \gamma P_0 \vec{\nabla}_{\parallel} u_{\parallel} = 0 \tag{4.89}$$

Taking the Fourier transform, this becomes

$$-i\omega P_1 + i\gamma P_0 k_{\parallel} u_{\parallel} - ik_y \frac{\phi_1}{B} \frac{\partial P_0}{\partial x} = 0$$

$$P_1 = \frac{\gamma P_0}{\omega} k_{\parallel} u_{\parallel} - \frac{k_y \phi_1}{\omega B} \frac{\partial P_0}{\partial x}$$

$$\frac{P_1}{P_0} = \frac{\gamma}{\omega} k_{\parallel} u_{\parallel} - \frac{k_y \phi_1}{\omega B} \frac{1}{P_0} \frac{\partial P_0}{\partial x} \tag{4.90}$$

Now, since $P_0 = n_0 k_B T_i$,

$$\frac{1}{P_0} \frac{\partial P_0}{\partial x} = \frac{1}{T_i} \frac{\partial T_i}{\partial x} + \frac{1}{n_0} \frac{\partial n_0}{\partial x} = \frac{1}{n_0} \frac{\partial n_0}{\partial x} (1 + \frac{n_0}{T_i} \frac{\partial T_i}{\partial x} \Big(\frac{\partial n_0}{\partial x} \Big)^{-1}) = -\frac{1}{L_n} (1 + \eta_i)$$

where $\eta_i \equiv \frac{\partial \ln T_i}{\partial \ln n_0}$ and $L_n \equiv -\frac{1}{n_0} \frac{\partial n_0}{\partial x}$. Note that these are both variables we've seen before, except now they are showing up in a different context. Plugging this result into equation 4.90 gives us

$$\frac{P_1}{P_0} = \frac{\gamma}{\omega} k_{\parallel} u_{\parallel} + \frac{k_y (1 + \eta_i) \phi_1}{\omega B L_n}$$

Using $\omega_{*i} = -\frac{k_y k_B T_i}{e B L_n}$ and $\omega_{*pi} = \omega_{*i}(1+\eta)$ (where the p stands for pressure), this becomes

$$\frac{P_1}{P_0} = \frac{\gamma}{\omega} k_{\parallel} u_{\parallel} - \frac{\omega_{*pi}}{\omega} \frac{e}{k_B T_e} \frac{T_e}{T_i} \phi_1$$

Plugging this into equation 4.87 gives us our parallel velocity $u_{\parallel}$.

$$u_{\parallel} = \frac{k_{\parallel} c_s^2}{2\omega} \Big[\frac{e\phi_1}{k_B T_e} \Big(1 - \frac{\omega_{*pi}}{\omega} \Big) + \frac{\gamma T_i}{\omega T_e} k_{\parallel} u_{\parallel} \Big]$$

$$u_{\parallel} = \frac{k_{\parallel} c_s^2}{2\omega} \Big[\frac{e\phi_1}{k_B T_e} \Big(1 - \frac{\omega_{*pi}}{\omega} \Big) \Big] \Big/ \Big[1 - \gamma \frac{k_{\parallel}^2 V_{Ti}^2}{2\omega^2} \Big] \tag{4.91}$$

We've now solved for each of the components of $\vec{u}_1$. This means that we can solve for $\vec{\nabla} \cdot \vec{u}_1$, and plug that into the equation for first-order ion density n_1, equation 4.83. We have

$$\vec{\nabla} \cdot \vec{u}_1 = \frac{\partial u_{1y}}{\partial y} + \frac{\partial u_{\parallel}}{\partial z} = -i \frac{\omega k_y^2 \phi_1}{\Omega_i B} + i \frac{k_{\parallel}^2 c_s^2}{2\omega} \Big[\frac{e\phi_1}{k_B T_e} \Big(1 - \frac{\omega_{*pi}}{\omega} \Big) \Big] \Big/ \Big[1 - \gamma \frac{k_{\parallel}^2 V_{Ti}^2}{2\omega^2} \Big]$$

where $V_{Ti}^2 = \frac{2k_BT_i}{m_i}$. Now, since $V_{Te} \gg \frac{\omega}{k_\parallel} \gg V_{Ti}$ for drift waves, we can eliminate the $\left[1-\gamma\frac{k_\parallel^2 V_{Ti}^2}{2\omega^2}\right]$ term in favor of just 1. Plugging this into our equation (4.83) for the ion density perturbation, we have

$$\left(\frac{n_1}{n_0}\right)_i = \frac{i}{\omega}\Big[\frac{-ik_y\phi_1}{BL_n} + i\frac{\omega k_y^2\phi_1}{\Omega_i B} - i\frac{k_\parallel^2 c_s^2}{2\omega}\Big[\frac{e\phi_1}{k_BT_e}\Big(1-\frac{\omega_{*pi}}{\omega}\Big)\Big]\Big]$$

$$\left(\frac{n_1}{n_0}\right)_i = \frac{e}{k_BT_e}\Big[\frac{k_yk_BT_e}{\omega eBL_n} - \frac{k_y^2k_BT_e}{m_i\Omega_i^2} + \frac{k_\parallel^2c_s^2}{2\omega^2}\Big(1-\frac{\omega_{*pi}}{\omega}\Big)\Big]\phi_1$$

$$\left(\frac{n_1}{n_0}\right)_i = \frac{e}{k_BT_e}\Big[\frac{\omega_{*e}}{\omega} - b_s + \frac{k_\parallel^2c_s^2}{2\omega^2}\Big(1-\frac{\omega_{*pi}}{\omega}\Big)\Big]\phi_1 \tag{4.92}$$

where $b_s = \frac{1}{2}\rho_s^2k_y^2$, $\rho_s^2 = \frac{c_s}{\Omega_i}$ and $\omega_{*e} = \frac{k_yk_BT_e}{eBL_n}$. Equation 4.92 is our ion response function. We've already calculated the electron response function, so using Gauss's law and the quasineutrality condition, we get (cancelling the factor of $\frac{n_0e^2}{k_BT_e}$)

$$\Big[\frac{\omega_{*e}}{\omega} - b_s + \frac{k_\parallel^2c_s^2}{2\omega^2}\Big(1-\frac{\omega_{*pi}}{\omega}\Big) - 1\Big]\phi_1 \tag{4.93}$$

This gives us our dispersion relation for the ion temperature gradient modes. Notice that if we didn't take into account ion temperature effects, we would recover the dispersion relation for drift waves because we could set $\omega_{*pi} = 0$. Remember how we solved this equation in the case that $\omega_{*pi} = 0$: we solved the equation perturbatively. First, we took $b_s = 0$ and $\frac{k_\parallel^2c_s^2}{2\omega^2} = 0$ which we justified on the assumption that the wavelength of the drift wave is very long, so k is small. This gave us $\omega \approx \omega_{*e}$ as a zeroth-order solution to the dispersion relation. Equation 4.93, on the other hand, doesn't lend itself to a nice perturbative solution when ω_{*pi} is not small. However, for sufficiently large ω_{*pi}, one of the solutions to this equation has an imaginary component which is *positive imaginary*, which (because we've assumed exponential dependence $e^{-i\omega t}$) implies that there is an exponentially growing mode. (is this statement totally true?) Rather than try to solve this equation perturbatively, let's solve it in the case where $\eta \gg 1$, so much so that $\frac{\omega_{*pi}}{\omega} \gg 1$ and $\left|\frac{k_\parallel^2c_s^2}{2\omega^2}\omega_{*pi}\right| \gg |\omega_{*e}|$. We also set $b_S = 0$ as usual, which corresponds to small k_y. In this case, the only terms we need to consider are the -1 and the $-\frac{k_\parallel^2c_s^2\omega_{*pi}}{2\omega^3}$ terms. This gives us

$$-\frac{k_\parallel^2c_s^2\omega_{*pi}}{2\omega^3} - 1 = 0$$

$$\omega^3 = -\frac{k_\parallel^2c_S^2\omega_{*pi}}{2}$$

Since $\omega_{*pi} \equiv \omega_{*i}(1+\eta)$ and

$$\omega_{*pi} \equiv -\frac{k_yk_BT_i}{eBL_n} < 0$$

then

$$\omega^3 > 0$$

This means that ω must have some imaginary component. The solution is

$$\omega = (|\omega_{*pi}|k_{\parallel}^2 c_s^2/2)^{1/3}(\frac{\sqrt{3}}{2}i - \frac{1}{2}) \tag{4.94}$$

We see that $(\frac{\sqrt{3}}{2}i - \frac{1}{2})^3 = 1$ as required, if we multiply it out.

$$(\frac{\sqrt{3}}{2}i - \frac{1}{2})(\frac{\sqrt{3}}{2}i - \frac{1}{2})(\frac{\sqrt{3}}{2}i - \frac{1}{2}) = (\frac{\sqrt{3}}{2}i - \frac{1}{2})(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) = 1$$

Question: how do we know it's not just 1 instead of $\sqrt{3}i/2 - \frac{1}{2}$?
Todo: write conclusion
Todo: figure out everything on the last page

4.4 Effect of Turbulent Fluctuations on Transport